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M20550 Calculus III Tutorial
Worksheet 2

1. Find an equation of the plane passes through the point $(1,1,-7)$ and perpendicular to the line $x = 1+4t$, $y = 1-t$, $z = -3$.

> ***Solution:*** To write an equation of a plane, we need one point on the plane and a normal vector (a vector that is perpendicular to the plane).
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> In this problem, we have the point $(1,1,-7)$ on the plane. Now, we need to find a normal vector. We know our plane is perpendicular to the line $x = 1+4t$, $y = 1-t$, $z = -3$. So, the parallel vector to this line, which is $\mathbf{v} = \langle 4,-1,0\rangle$, can be used as the normal vector to our plane.
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> Finally, an equation of the plane with normal vector $\langle 4,-1,0\rangle$ passing through $(1,1,-7)$ is given by
>
> $$\begin{aligned} \langle 4,-1,0\rangle \cdot \langle x,y,z\rangle &= \langle 4,-1,0\rangle \cdot \langle 1,1,-7\rangle \\ \implies 4x - y &= 3. \end{aligned}$$

2. Let ℓ be the line of intersection of the planes given by equations $x - y = 1$ and $x - z = 1$. Find an equation for ℓ in the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

> ***Solution:*** To write an equation of the line ℓ, we need to find one point on ℓ and a parallel vector to ℓ.
>
> Since ℓ is the line of intersection of two planes, to find a point on ℓ, we need to find a point that is contained in both planes. A point on both planes can be found by setting $x = 1$, so $y = z = 0$. And we get the point $(1,0,0)$ on ℓ.
>
> A normal vector for the first plane is $\langle 1,-1,0\rangle$ and a normal vector for the second plane is $\langle 1,0,-1\rangle$. A parallel vector of ℓ is a vector perpendicular to the normal vectors of both planes. Thus, a parallel vector of ℓ is given by
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> $$\langle 1,-1,0\rangle \times \langle 1,0,-1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1,1,1\rangle .$$
>
> Hence, the vector equation of ℓ is
>
> $$\mathbf{r}(t) = \langle 1,0,0\rangle + t\,\langle 1,1,1\rangle .$$

Another way to solve this problem is just to consider the following, ℓ is the set of points that satisfy both

$$\begin{aligned} x - y &= 1 \\ x - z &= 1 \end{aligned}$$

which is equivalent to the set of points which satisfy

$$x - 1 = y = z$$

which is the cartesian equation for the line ℓ, the to go from the cartesian equation to the vector equation, we just set

$$\begin{aligned} x - 1 &= t \\ y &= t \\ z &= t \end{aligned}$$

and this system of equations is equivalent to the system

$$\begin{aligned} x &= 1 + t \\ y &= t \\ z &= t \end{aligned}$$

and this gives us that a vector equation for ℓ is given by;

$$\mathbf{r}(t) = (1, 0, 0) + t\langle 1, 1, 1\rangle.$$

3. How many times does a particle traveling along the curve $\mathbf{r}(t) = \langle t^2 + 1, 2t^2 - 1, 2 - 3t^2\rangle$ hit the plane
$2x + 2y + 3z = 3$? What is the point(s) of intersection?

Solution: (a) We have $\mathbf{r}(t) = \langle t^2 + 1, 2t^2 - 1, 2 - 3t^2\rangle$. So the x, y, z-coordinates of the particle are given by:

$$x = t^2 + 1, \qquad y = 2t^2 - 1, \qquad z = 2 - 3t^2.$$

If the particle hits the plane, the x, y, z-coordinates of the particle have to satisfy the equation $2x + 2y + 3z = 3$. Thus, we get the equation

$$\begin{aligned} 2(t^2+1) + 2(2t^2-1) + 3(2-3t^2) &= 3 \\ 2t^2 + 2 + 4t^2 - 2 + 6 - 9t^2 &= 3 \\ -3t^2 + 6 &= 3 \\ t^2 &= 1 \\ t &= 1 \quad \text{or} \quad t = -1 \end{aligned}$$

Thus, the particle hits the plane twice. And with $t = 1$, we get $x = 1^2 + 1 = 2$, $y = 2(1)^2 - 1 = 1$, $z = 2 - 3(1)^2 = -1 \implies (2, 1, -1)$.
With $t = -1$, $x = (-1)^2 + 1 = 2$, $y = 2(-1)^2 - 1 = 1$, $z = 2 - 3(-1)^2 = -1 \implies (2, 1, -1)$. So, we only have one point of intersection, that is $(2, 1, -1)$.

4. Let P be a plane with normal vector $\langle -2, 2, 1 \rangle$ passing through the point $(1, 1, 1)$. Find the distance from the point $(1, 2, -5)$ to the plane P.

Solution: Let's make a vector $\mathbf{b}$ from the point $(1, 1, 1)$ to the point $(1, 2, -5)$:

$$\mathbf{b} = \langle 1-1, 2-1, -5-1 \rangle = \langle 0, 1, -6 \rangle.$$

Then, the distance D from the point $(1, 2, -5)$ to the plane P is given by

$$D = |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|\langle -2, 2, 1 \rangle \cdot \langle 0, 1, -6 \rangle|}{|\langle -2, 2, 1 \rangle|} = \frac{|-4|}{\sqrt{(-2)^2 + 2^2 + 1^2}} = \frac{4}{3}.$$

5. Find an equation of the plane that passes through the point $(1, 2, 3)$ and contains the line $\frac{1}{3}x = y - 1 = 2 - z$.

Solution: For this problem, in order to find a normal vector of the plane, we first need to find two vectors on the plane then take their cross product.

One vector that lies on the plane is a parallel vector of the line $\frac{1}{3}x = y - 1 = 2 - z$ (because this line is contained in the plane). Note that $\frac{1}{3}x = y - 1 = 2 - z \iff \frac{x-0}{3} = \frac{y-1}{1} = \frac{z-2}{-1}$. So, a parallel vector of this line is $\mathbf{v_1} = \langle 3, 1, -1 \rangle$. Thus, we have $\mathbf{v_1} = \langle 3, 1, -1 \rangle$ lies on the plane.

To get another vector on the plane, we take one point on the line and make a vector with the point on the plane $(1, 2, 3)$. One point on the line $\dfrac{x-0}{3} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$ is $(0, 1, 2)$. So, we get the second vector $\mathbf{v_2}$ on the plane, $\mathbf{v_2} = \langle 1-0, 2-1, 3-2\rangle = \langle 1, 1, 1\rangle$.

Then, a normal vector is given by

$$\mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \langle 2, -4, 2\rangle .$$

So, the equation of the required plane is

$$\begin{aligned} \langle 2, -4, 2\rangle \cdot \langle x, y, z\rangle &= \langle 2, -4, 2\rangle \cdot \langle 1, 2, 3\rangle \\ \implies 2x - 4y + 2z &= 0 \\ \implies x - 2y + z &= 0 \end{aligned}$$

6. Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2 = 9$ and the plane $x + y - z = 5$.

Solution: To find a vector function that represents the curve of intersection, we need to be able to describe x, y, z in terms of t for this curve.

On the xy-plane, $x^2 + y^2 = 9$ represents a circle centers at the origin with radius 3. So, we can write the parametric equations for this circle as follows:

$$x = 3\cos t, \qquad y = 3\sin t, \qquad 0 \le t \le 2\pi.$$

And from the equation of the plane, we get

$$z = x + y - 5 \implies z = 3\cos t + 3\sin t - 5, \qquad 0 \le t \le 2\pi.$$

So, a vector function that represents the curve of intersection is given by

$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + (3\cos t + 3\sin t - 5)\,\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

7. Give a vector valued function that describes the position of a particle that starts at the point $(0, 1)$ at time $t = 0$ and then moves along the unit circle in the xy-plane clockwise.

Solution: Observe that if we take our usual parametrization of the unit circle, $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, and reflect through the line $y = x$, we get the motion we wish to describe. Then recall that reflection through the line $y = x$ is given by the map $(a, b) \mapsto (b, a)$. So one solution is the function

$$\boldsymbol{\phi}(t) = \langle \sin(t), \cos(t) \rangle.$$

Note: there are many solutions to this problem, $\boldsymbol{\phi}(ct) = \langle \sin(ct), \cos(ct) \rangle$ for any positive value of c would work also. (These just represent the particle moving at different speeds.)