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M20550 Calculus III Tutorial
Worksheet 5

1. Find $\dfrac{dz}{dt}$ when $t = 2$, where $z = x^2 + y^2 - 2xy$, $x = \ln(t-1)$ and $y = e^{-t}$.

> ***Solution:*** We have $z = z\big(x(t), y(t)\big)$. So, by the chain rule, we obtain
>
> $$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2x-2y)\left(\frac{1}{t-1}\right) + (2y-2x)e^{-t}(-1) \\ &= \big(2\ln(t-1) - 2e^{-t}\big)\left(\frac{1}{t-1}\right) - \big(2e^{-t} - 2\ln(t-1)\big)\, e^{-t}. \end{aligned}$$
>
> Hence,
>
> $$\begin{aligned} \left.\frac{dz}{dt}\right|_{t=2} &= \big(2\ln(2-1) - 2e^{-2}\big)\left(\frac{1}{2-1}\right) - \big(2e^{-2} - 2\ln(2-1)\big)\, e^{-2} \\ &= (0 - 2e^{-2}) \cdot 1 - (2e^{-2} - 0)e^{-2} \\ &= -2e^{-2} - 2e^{-4}. \end{aligned}$$

2. Let $r = r(x, y)$, $x = x(s, t)$, and $y = y(t)$. Given that

$$\begin{array}{lll} x(1,0) = 2, & x_s(1,0) = -1, & x_t(1,0) = 7, \\ y(0) = 3, & y(1) = 0 & y'(0) = 4, \\ r(2,3) = -1, & r_x(2,3) = 3, & r_y(2,3) = 5, \\ r_x(1,0) = 6, & r_y(1,0) = -2, & \end{array}$$

calculate $\dfrac{\partial r}{\partial t}$ at $s = 1, t = 0$.

> ***Solution:*** We have $r = \big(x(s,t), y(t)\big)$. So, from the chain rule, we get
>
> $$\begin{aligned} \frac{\partial r}{\partial t} &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial r}{\partial y}\frac{dy}{dt} \\ &= r_x x_t + r_y y' \\ &= r_x(x,y)x_t(s,t) + r_y(x,y)y'(t). \end{aligned}$$

When $s = 1$ and $t = 0$, we have $x = x(1,0) = 2$ and $y = y(0) = 3$. So,

$$\begin{aligned} \left.\frac{\partial r}{\partial t}\right|_{s=1,\, t=0} &= r_x(2,3)x_t(1,0) + r_y(2,3)y'(0) \\ &= (3)(7) + (5)(4) \\ &= 41. \end{aligned}$$

3. (a) Let $f(x, y, z) = x^2 - yz$. If $\mathbf{v} = \langle 1, 1, 0\rangle$, find the directional derivative of f in the direction of $\mathbf{v}$ at the point $(1, 2, 3)$.

(b) Interpret your result in part (a) by filling in the blanks and circling the correct word of the statement below:

At the point ____________, the value of the function f is *increasing / decreasing* at the rate of ____________ as we move in the direction given by the vector ____________.

Solution: (a) The directional derivative of f in the direction of $\mathbf{v}$ at the point $(1, 2, 3)$, denote $D_{\mathbf{u}}f(1, 2, 3)$ where $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$, is given by

$$D_{\mathbf{u}}f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \mathbf{u}$$

First,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 1, 1, 0\rangle}{\sqrt{1^2 + 1^2 + 0^2}} = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle.$$

Secondly, the gradient of f is given by:

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \langle 2x, -z, -y\rangle \\ \implies \nabla f(1, 2, 3) &= \langle 2, -3, -2\rangle. \end{aligned}$$

So, now

$$\begin{aligned} D_{\mathbf{u}}f(1, 2, 3) &= \nabla f(1, 2, 3) \cdot \mathbf{u} \\ &= \langle 2, -3, -2\rangle \cdot \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle \\ &= \frac{1}{\sqrt{2}}\langle 2, -3, -2\rangle \cdot \langle 1, 1, 0\rangle \\ &= \frac{1}{\sqrt{2}}(2 - 3) \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

(b) At the point $\underline{(1,2,3)}$, the value of the function f is $\boxed{\text{decreasing}}$ at the rate of $\underline{1/\sqrt{2}}$ as we move in the direction given by the vector $\underline{\langle 1,1,0\rangle}$.

4. Let $f(x,y) = \ln(xy)$. Find the maximum rate of change of f at $(1,2)$ and the direction in which it occurs.

Solution: It is a fact that f changes the fastest in the direction of its gradient vector and the maximum rate of change is the magnitude of the gradient vector.

With $f(x,y) = \ln(xy)$, we first compute $\nabla f(1,2)$:

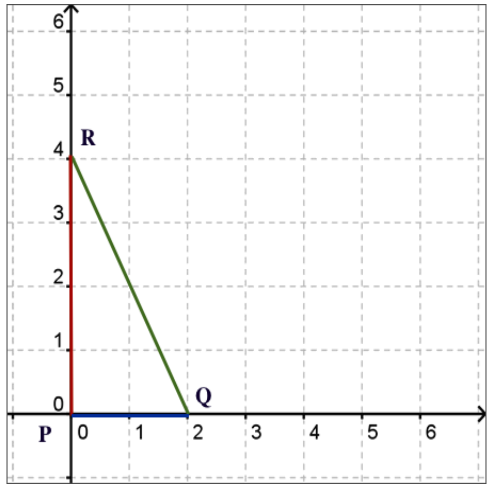
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{y}{xy}, \frac{x}{xy} \right\rangle = \left\langle \frac{1}{x}, \frac{1}{y} \right\rangle$$
$$\implies \nabla f(1,2) = \left\langle 1, \frac{1}{2} \right\rangle.$$

$$\implies |\nabla f(1,2)| = \left| \left\langle 1, \frac{1}{2} \right\rangle \right| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}.$$

So, the maximum rate of change of f at $(1,2)$ is $\dfrac{\sqrt{5}}{2}$ and the direction in which it occurs is $\left\langle 1, \dfrac{1}{2} \right\rangle$.

5. Identify the absolute maximum and absolute minimum values attained by $g(x,y) = x^2y - 2x^2$ within the triangle T bounded by the points $P(0,0)$, $Q(2,0)$, and $R(0,4)$.

Solution: The picture for the triangle T:



First, we find all critical points in the interior of the triangle:

$$\begin{cases} g_x(x,y) = 2xy - 4x = 0 & (1) \\ g_y(x,y) = x^2 = 0 & (2) \end{cases}$$

Equation (2) tells us that x must be zero. And when $x = 0$, equation (1) is true automatically giving us the points $(0, y)$ for $0 \le y \le 4$ are the solutions of this system of equations. So, all the critical points are exactly the boundary PR of the triangle. So, we get no critical point inside the triangle. We move on to analyze the boundaries.

On the boundary PR, we have $x = 0$ and $0 \le y \le 4$. And, $g(0, y) = 0$.

On the boundary PQ, we have $0 \le x \le 2$ and $y = 0$. And, $g(x, 0) = -2x^2$. The graph of $-2x^2$ is a parabola concaves downward. So, $g(x, 0) = -2x^2$ with $0 \le x \le 2$ attains a maximum value of 0 when $x = 0$ and a minimum value of -8 when $x = 2$.

On the boundary QR, we have $y = -2x + 4$ with $0 \le x \le 2$. And, $g(x, -2x + 4) = x^2(-2x + 4) - 2x^2 = -2x^3 + 2x^2$, for $0 \le x \le 2$. The critical numbers of $-2x^3 + 2x^2$ for $0 \le x \le 2$ are $x = 0$ and $x = \frac{2}{3}$. So, g has a minimum of 0 at $x = 0$ and a maximum of $\frac{8}{27}$ at $x = \frac{2}{3}$, $y = \frac{8}{3}$ on this boundary.

Here is a summary of the results:

(x, y)	$g(x, y)$
$(0, y)$	0
$(2, 0)$	-8
$\left(\frac{2}{3}, \frac{8}{3}\right)$	$\frac{8}{27}$

So, we conclude that on the whole triangle (including boundaries), the function has an absolute maximum of $\frac{8}{27}$ at $\left(\frac{2}{3}, \frac{8}{3}\right)$ and an absolute minimum of -8 at $(2, 0)$.

6. Identify the absolute maximum and absolute minimum values attained by $z = 4x^2 - y^2 + 1$ on the region $R = \{(x, y) \mid 4x^2 + y^2 \leq 16\}$.

Solution: First, we find the critical points in the interior of the region R. We have

$$\begin{cases} z_x(x, y) = 8x = 0 & \implies x = 0 \\ z_y(x, y) = -2y = 0 & \implies y = 0 \end{cases}$$

So, the only critical point inside R is $(0, 0)$.

Next, we want to find the extreme values of z on the **boundary** $4x^2 + y^2 = 16$. One way of doing this is to use the method of Lagrange Multipliers. In this language, we want to find the extrema of $z = 4x^2 - y^2 + 1$ subject to the constraint $g(x, y) = 4x^2 + y^2 = 16$. We have $\nabla z = \lambda \nabla g$ for some constant λ. So, we get the system of equations:

$$\begin{cases} 8x = \lambda 8x & (1) \\ -2y = \lambda 2y & (2) \\ 4x^2 + y^2 = 16 & (3) \end{cases}$$

Equation (1) $\Leftrightarrow 8x(1 - \lambda) = 0 \implies x = 0$ or $\lambda = 1$.

- If $x = 0$, then from equation (3) we get $y = \pm 4$. And so we get $(0, \pm 4)$ as the points of interest.
- If $\lambda = 1$, then from equation (2) we get $y = 0$. With $y = 0$, equation (3) gives $x = \pm 2$. So, the points of interest are $(\pm 2, 0)$.

Finally, let's compute the values of z at all the points we found:

(x, y)	$z = 4x^2 - y^2 + 1$
$(0, 0)$	1
$(0, -4)$	-15
$(0, 4)$	-15
$(-2, 0)$	17
$(2, 0)$	17

In conclusion, the absolute maximum value of z is 17 and it occurs at the points $(-2, 0)$ and $(2, 0)$. The absolute minimum value of z is -15 and it occurs at the points $(0, -4)$ and $(0, 4)$.

7. Find all points on the surface $z = x^2 - y^3$ where the tangent plane is parallel to the plane $x + 3y + z = 0$.

Solution: First, rewrite $z = x^2 - y^3$ into the level surface $F(x, y, z) = x^2 - y^3 - z = 0$ then $\nabla F(x, y, z) = \langle 2x, -3y^2, -1\rangle$ gives a normal vector to the tangent plane at any point (x, y, z) on the surface.

We want to find a point (x, y, z) such that the tangent plane is parallel to the plane $x + 3y + z = 0$; so we want to find x, y, z such that $\nabla F(x, y, z) = k\langle 1, 3, 1\rangle$, for some scalar k. We have $\langle 2x, -3y^2, -1\rangle = k\langle 1, 3, 1\rangle$ implies

$$\begin{cases} 2x &= k \\ -3y^2 &= 3k \\ -1 &= k \end{cases}$$

So, $k = -1$ (no other k works for this system of equations). Thus, we get $2x = -1 \implies x = -\frac{1}{2}$, and $-3y^2 = -3 \implies y = \pm 1$. Now we need to find z. Remember the point (x, y, z) we are looking for is on the surface $z = x^2 - y^3$.

So then with $x = -\frac{1}{2}$ and $y = 1$, we get $z = \left(-\frac{1}{2}\right)^2 - (1)^3 = -\frac{3}{4}$.

And with $x = -\frac{1}{2}$ and $y = -1$, we get $z = \left(-\frac{1}{2}\right)^2 - (-1)^3 = \frac{5}{4}$.

So, at the points $\left(-\frac{1}{2}, 1, -\frac{3}{4}\right)$ and $\left(-\frac{1}{2}, -1, \frac{5}{4}\right)$, the tangent plane to the surface $z = x^2 - y^3$ is is parallel to the plane $x + 3y + z = 0$.

More Practice Problems:

8. *(This usually is a challenging problem to students)* Find **all** points at which the direction of fastest change of the function
$f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

Solution: We know the direction of fastest change of f at a point (x, y) is given by the direction of $\nabla f(x, y) = \langle 2x-2, 2y-4\rangle$. So, we want to find all pairs (x, y) such that $\langle 2x-2, 2y-4\rangle = k\langle 1, 1\rangle$ for any constant k. We obtain the system of equations

$$\begin{cases} 2x-2 &= k \\ 2y-4 &= k \end{cases}$$

Then, $2x-2 = 2y-4 \implies y = x+1$. Thus, all the wanted pairs (x, y) are $(x, x+1)$, where x admits any value in the domain. This is exactly all the points on the line $y = x+1$ in the domain of f.

9. If $h = x^2 + y^2 + z^2$ and f is a differentiable function of two variables that satisfies the equation

$$y \cos f(x, y) + f(x, y) \cos x = 0$$

at every point (x, y) in its domain, find

$$\frac{\partial(h(x, y, f(x, y)))}{\partial x}.$$

Solution: So,

$$\frac{\partial h(x, y, f(x, y))}{\partial x} = \frac{\partial[x^2 + y^2 + f(x, y)^2]}{\partial x} = 2x + 2f(x, y)\frac{\partial f(x, y)}{\partial x}$$

For the following calculation, we let z stand for $f(x, y)$, just to make the notation easier to handle. To find $\frac{\partial f(x, y)}{\partial x}$, we use implicit differentiation:

$$\begin{aligned} y \cos z + z \cos x &= 0 \\ \frac{\partial}{\partial x}\left[y \cos z + z \cos x\right] &= \frac{\partial}{\partial x}\left[0\right] \\ -y \sin z \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \cos x - z \sin x &= 0 \\ \frac{\partial z}{\partial x} (\cos x - y \sin z) &= z \sin x \\ \frac{\partial z}{\partial x} &= \frac{z \sin x}{\cos x - y \sin z} \end{aligned}$$

Therefore,

$$\frac{\partial h(x, y, f(x, y))}{\partial x} = 2x + 2f(x, y)\left(\frac{f(x, y) \sin x}{\cos x - y \sin(f(x, y))}\right)$$

$$\implies \frac{\partial h(x, y, f(x, y))}{\partial x} = 2x + \frac{2f(x, y)^2 \sin x}{\cos x - y \sin f(x, y)}.$$

10. A cylinder containing an incompressible fluid is being squeezed from both ends. If the length of the cylinder is *decreasing* at a rate of 3m/s, calculate the rate at which the radius is changing when the radius is 2m and the length is 1m. (Note: An incompressible fluid is a fluid whose volume does not change.)

Solution: Let V be the volume of the cylinder, r be the radius of the cylinder, and l be its length. Then, $V = \pi r^2 l$. So, $V = V\big(r(t), l(t)\big)$.

By assumptions, we have $\frac{dl}{dt} = -3$ and incompressibility of the fluid implies $\frac{dV}{dt} = 0$. We want to find $\frac{dr}{dt}$ at the instant when $r = 2$ and $l = 1$. We have

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}\left[\pi r^2 l\right]\\ 0 &= 2\pi r l\frac{dr}{dt} + \pi r^2\frac{dl}{dt}. \quad \text{And we know } \frac{dl}{dt} = -3;\ \text{so}\\ 0 &= 2\pi r l\frac{dr}{dt} - 3\pi r^2\\ \frac{dr}{dt} &= \frac{3r}{2l}.\end{aligned}$$

Hence, when $r = 2, l = 1$, we get $\frac{dr}{dt} = \frac{3\cdot 2}{2\cdot 1} = 3\text{m/s}$.