Name: SOLUTIONS Date: 03/05/2020

M20550 Calculus III Tutorial
Worksheet 6

1. *(The D-formula)* Find the local maximum and the local minimum value(s) and saddle point(s) of the function $z = x^3 + y^3 - 3xy + 1$.

Solution: First, let's find all the critical points of $z = x^3 + y^3 - 3xy + 1$:

$$\begin{cases} z_x(x,y) = 3x^2 - 3y = 0 \implies y = x^2 & (1) \\ z_y(x,y) = 3y^2 - 3x = 0 & (2) \end{cases}$$

With $y = x^2$, equation (2) becomes $3x^4 - 3x = 0 \implies 3x(x^3 - 1) = 0 \implies x = 0$ or $x = 1$. Thus, all the critical points are $(0,0)$ and $(1,1)$.

Now, we will use the Second Derivative Test to test each critical point. We want to compute

$$D(x,y) = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = z_{xx}z_{yy} - z_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.$$

And we have

$$D(0,0) = -9 < 0 \implies (0,0) \text{ is a saddle point.}$$

$$D(1,1) = 36 - 9 > 0 \text{ and } z_{xx}(1,1) = 6 > 0 \implies z(1,1) \text{ is a local minimum.}$$

In conclusion, the local minimum value of z is $z(1,1) = 1^3 + 1^3 - 3(1)(1) + 1 = 0$. And $(0,0)$ is the saddle point of z, i.e. $z(0,0)$ is neither a local minimum nor local maximum.

2. Evaluate the double integral $\iint_R (4 - 2y)\,dA$, for $R = [0,1] \times [0,1]$, by identifying it as the volume of a solid.

Solution: Notice that $z = f(x,y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0,1] \times [0,1] \times [0,4]$ which lies below the plane $z = 4 - 2y$. We can compute this by taking the areas of the rectangular part and the triangular part, and multiplying their sum by the "depth" in the x-direction:

$$\iint_R (4 - 2y)\,dA = \left((1)(2) + \frac{(1)(2)}{2} \right)(1) = 3$$

3. Evaluate the iterated integral.

   (a) $\int_0^2 \int_0^\pi r \sin^2\theta \; d\theta dr$

   Solution: Since the region of integration is rectangular and the function is separable in θ and r, we can split it as a product of two integrals

   $$\int_0^2 r\; dr \cdot \int_0^\pi \sin^2\theta \; d\theta = 2 \cdot \int_0^\pi \frac{1}{2}(1 - \cos 2\theta)\; d\theta = \pi$$

   (b) $\iint_R ye^{-xy} dA$ on $R = [0,2] \times [0,3]$

   Solution: Notice that the region is rectangular, so the order of integration doesn't matter. However, we cannot separate this as a product of two integrals, since x and y are mixed variables in the function (we can't write it as a product of two functions $f(x)$ times $g(y)$).
   We could try to integrate with respect to y first, but that would require integration by parts. It turns out it is easier to start with x instead:

   $$\int_0^3 \int_0^2 ye^{-xy} dxdy = \int_0^3 [-e^{-xy}]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)dy = \frac{1}{2}e^{-6} + \frac{5}{2}$$

4. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

   Solution: The cylinder intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $[0,4] \times [0,5]$ in the xy-plane. Then

   $$V = \int_0^5 \int_0^4 (16 - x^2)dxdy = \int_0^5 dy \int_0^4 (16 - x^2)dx = 5\left[16x - \frac{1}{3}x^3\right]_0^4 = \frac{640}{3}$$

5. *(Double integrals over general regions)* Evaluate the following integrals:

   (a) $\iint_D xydA$, D is enclosed by the curves $y = x^2$, $y = 3x$;

   (b) $\iint_D ydA$, D is bounded by $y = x - 2$, $x = y^2$.

Solution:

(a) Let's find the intersection of the two curves:

$$x^2 = 3x \quad \Leftrightarrow \quad x = 0,\ x = 3;$$

Since the curves are expressed in terms of x, we see that it's easier to first integrate with respect to y variable. To determine which curve lies above another, one can, for instance, sketch the region.

Let's compute the integral:

$$\iint_D xyA = \int_0^3 dx \int_{x^2}^{3x} xydy = \int_0^3 x\, \frac{y^2}{2}\bigg|_{3x}^{x^2} dx = \frac{1}{2}\int_0^3 x(x^4 - 9x^2)dx =$$

$$= \frac{1}{2}\int_0^3 x^5 - 9x^3\, dx = \frac{1}{2}\left(\frac{x^6}{6}\bigg|_0^3 - 9\,\frac{x^4}{4}\bigg|_0^3 \right) = \frac{1}{2}\left(\frac{3^6}{6} - 9\frac{3^4}{4}\right) = \frac{3^6}{4}\left(\frac{1}{3} - \frac{1}{2}\right) =$$

$$= -\frac{3^5}{8} = -\frac{243}{8}.$$

(b) Find the intersection of the curves:

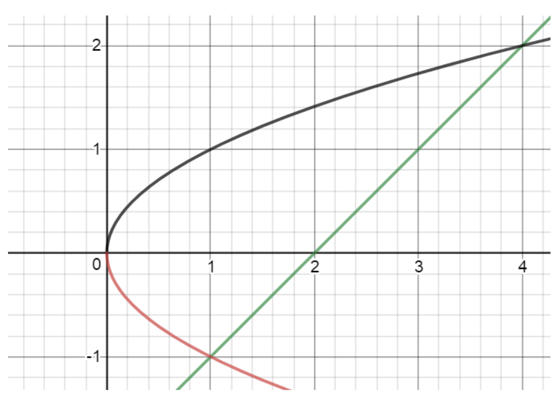
$$y + 2 = y^2 \quad \Leftrightarrow \quad y^2 - y - 2 = 0 \quad \Leftrightarrow \quad y = 2,\ y = -1.$$

These curves are conveniently expressed as functions of y, so we can take the boundaries of the inner integrals as functions of y and integrate first with respect to x:

$$\iint_D ydA = \int_{-1}^2 dy \int_{y^2}^{y+2} ydx = \int_{-1}^2 y(y + 2 - y^2)dy =$$

$$= \left(\frac{y^3}{3} + y^2 - \frac{y^4}{4}\right)\bigg|_{-1}^2 = \left(\frac{8}{2} + 4 - \frac{16}{4}\right) - \left(\frac{-1}{3} + 1 - \frac{1}{4}\right) = \frac{43}{12}.$$

Remark. To understand which curve is the upper bound and which is the lower, sketch the region over which you integrate. In this case, the region is



6. *(Fubini's theorem)* Change the order of integration in the following integrals:

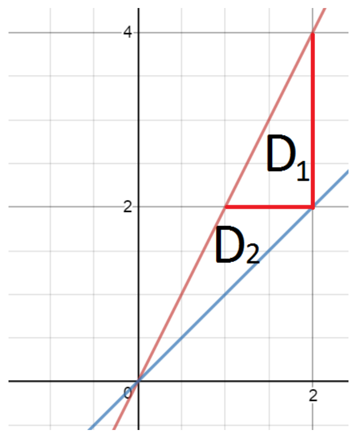
   (a) $\int_0^2 dx \int_x^{2x} f(x,y)dy$;

   (b) $\int_{-6}^2 dx \int_{\frac{x^2}{4}-1}^{2-x} f(x,y)dy$;

   Hint: in the second case you may need to sketch the region and to split the integral into two integrals over smaller regions.

Solution:

(a) Sketch the region:



We see that when we change the order of integration, the upper curve is piecewise. So, we split the region into two parts D_1 and D_2 so that the upper curves are not piecewise (see the remark to the next bullet for an alternative approach). Then

$$\int_0^2 dx \int_x^{2x} f(x,y)dy = \iint_{D_1 \cup D_2} f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA.$$

The first integral is

$$\iint_{D_1} f(x,y)dA = \int_2^4 dy \int_{y/2}^2 f(x,y)dx;$$

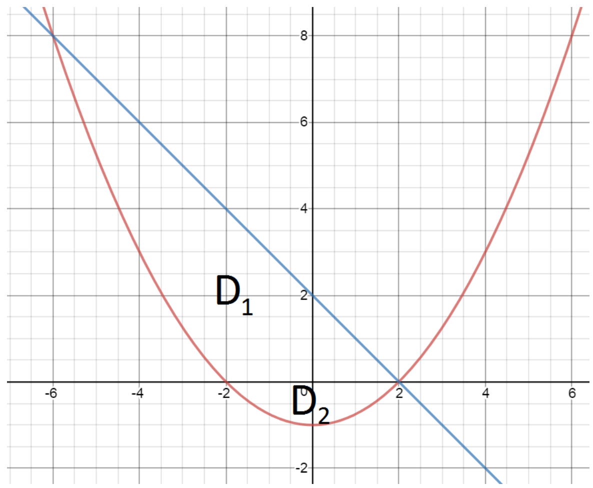
The second integral is

$$\iint_{D_2} f(x,y)dA = \int_0^2 dy \int_{y/2}^y f(x,y)dx.$$

Thus

$$\int_0^2 dx \int_x^{2x} f(x,y)dy = \int_2^4 dy \int_{y/2}^2 f(x,y)dx + \int_0^2 dy \int_{y/2}^y f(x,y)dx.$$

(b) Let's sketch the region:



Where D_1 is the region above x-axis and D_2 is the region below x-axis. Let's write the integral over the whole region $D_1 \cup D_2$ as

$$\iint_{D_1 \cup D_2} f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA.$$

If we solve for x in $y = x^2/4 - 1$, the we obtain two branches of the parabola:

$$x = \sqrt{4(y+1)} \quad \text{and} \quad x = -\sqrt{4(y+1)}.$$

Then the integral over D_1 can be written as

$$\iint_{D_1} f(x,y)dA = \int_0^8 dy \int_{-\sqrt{4(y+1)}}^{y+2} f(x,y)dy,$$

where the number 8 is obtained from plugging into $x^2/4 - 4$ the former lower bound -6. The inner boundaries of the integral over D_2 are the two branches of the parabola:

$$\iint_{D_2} f(x,y)dA = \int_{-1}^0 dy \int_{-\sqrt{4(y+1)}}^{\sqrt{4(y+1)}} f(x,y)dx.$$

Thus the resulting integral over $D_1 \cup D_2$ is

$$\iint_{D_1 \cup D_2} f(x,y)dA = \int_0^8 dy \int_{-\sqrt{4(y+1)}}^{y+2} f(x,y)dy + \int_{-1}^0 dy \int_{-\sqrt{4(y+1)}}^{\sqrt{4(y+1)}} f(x,y)dx.$$

Remark. Instead of splitting the region into two parts, we could also define a function

$$g(y) := \begin{cases} y+2, \ \ y \geq 0, \\ \sqrt{4(y+1)}, \quad -1 \leq y \leq 0. \end{cases}$$

In other words, we could split the boundary itself. Then the integral could be written as
$$\int_{-6}^{2} dx \int_{x^2/4-1}^{2-x} f(x,y)dy = \int_{-1}^{8} \int_{-\sqrt{4(y+1)}}^{g(y)} f(x,y)dy.$$
However, if we were given an actual function $f(x,y)$ and were asked to compute the integral, we would have to split the integral again into integrals over the two regions so that $g(y)$ is given by an explicit formula.

7. *(Optional: Lagrange multipliers with two constraints)* Find the maximum value of the function $f(x,y,z) = x + 2y$ on the curve of intersection of the plane $x + y + z = 1$ and the cylinder $y^2 + z^2 = 4$.

Solution: Basically, the problem asks to maximize f subject to two constraints:

$$g(x,y,z) = x + y + z = 1$$
$$h(x,y,z) = y^2 + z^2 = 4$$

We'll do this problem by the method of Lagrange Multipliers: First compute

$$\nabla f(x,y,z) = \langle 1, 2, 0\rangle$$
$$\nabla g(x,y,z) = \langle 1, 1, 1\rangle$$
$$\nabla h(x,y,z) = \langle 0, 2y, 2z\rangle$$

We know $\nabla f = \lambda \nabla g + \mu \nabla h$ for some scalars λ, μ. So, along with the two constraints, we have the following system of equations:

$$\begin{cases} 1 & = \lambda & (1) \\ 2 & = \lambda + 2\mu y & (2) \\ 0 & = \lambda + 2\mu z & (3) \\ x + y + z & = 1 & (4) \\ y^2 + z^2 & = 4 & (5) \end{cases}$$

We get $\lambda = 1$ from equation (1). Putting this into equations (2) and (3), we get

$$\begin{cases} 1 & = 2\mu y \\ -1 & = 2\mu z. \end{cases}$$

Adding these two equations, we get $2\mu y + 2\mu z = 0 \implies 2\mu(y + z) = 0$. So, $\mu = 0$ or $y = -z$.

If $\underline{\mu = 0}$, then from equation (2), we have $2 = 1$, a contradiction. So, $\mu \neq 0$.

If $\underline{y = -z}$, then equation (5) yields $2z^2 = 4 \implies z = \pm\sqrt{2}$. So then $y = \mp\sqrt{2}$. And from equation (4), $x = 1 - y - z$. So, $x = 1 - (-\sqrt{2}) - \sqrt{2} = 1$ or $x = 1 - \sqrt{2} - (-\sqrt{2}) = 1$.
So, we obtain the points $(1, -\sqrt{2}, \sqrt{2})$ and $(1, \sqrt{2}, -\sqrt{2})$.

So then,

$$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$$
$$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}.$$

Thus, the maximum value of f is $1 + 2\sqrt{2}$ on the curve of intersection.