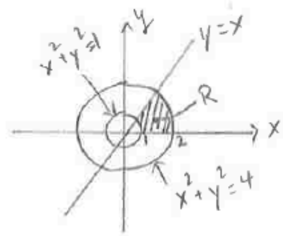
Name: Date: 26/03/2020

M20550 Calculus III Tutorial
Worksheet 7

1. Evaluate the given integral.
$$\iint_R \arctan\left(\frac{y}{x}\right)\, dA$$
where $R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.

Solution:



Given the geometry of region R, it's best to compute the double integral using polar coordinates.

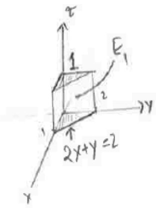
In polar, we know $dA = r\,dr\,d\theta$ and
$$\arctan\left(\frac{y}{x}\right) = \arctan(\tan\theta) = \theta \quad \left(\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$$

From the picture of the region R, we have $1 \leq r \leq 2$. To find the upper bound for θ, we need to find θ in (I) quad. such that $y = x$. With $y = x$, we have $r\sin\theta = r\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$ for θ in (I) quad. So, $0 \leq \theta \leq \frac{\pi}{4}$.

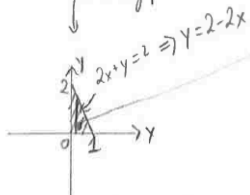
Thus,
$$\iint_R \arctan\left(\frac{y}{x}\right) dA = \int_0^{\pi/4}\int_1^2 \theta\, r\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{2}r^2\theta\Big|_{r=1}^{r=2} d\theta = \int_0^{\pi/4} \frac{3}{2}\theta\, d\theta = \frac{3}{2}\cdot\frac{1}{2}\theta^2\Big|_0^{\pi/4} = \boxed{\frac{3}{64}\pi^2}$$

2. (a) Let E_1 be the solid that lies under the plane $z = 1$ and above the region in the xy-plane bounded by $x = 0$, $y = 0$, and $2x + y = 2$. Write the triple integral $\iiint_{E_1} xz\, dV$ but do not evaluate it.

Solution:



in the xy-plane



With the order $dy\,dx$:

$0 \le y \le 2-2x$

$0 \le x \le 1$

We'll use rectangular coordinates to write $\iiint_{E_1} xz\,dV$.

$$\iiint_{E_1} xz\,dV = \iint_R \left(\int_{z=0}^{z=1} xz\,dz \right) dA$$

(now write $dA = dy\,dx$, and the limits for y and x comes from the picture of R)

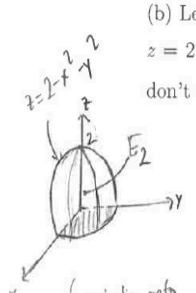
$$= \int_{x=0}^{x=1} \int_{y=0}^{y=2-2x} \int_{z=0}^{z=1} xz\,dz\,dy\,dx$$

(Another answer is $\int_{y=0}^{y=2} \int_{x=0}^{x=\frac{2-y}{2}} \int_{z=0}^{z=1} xz\,dz\,dx\,dy$)

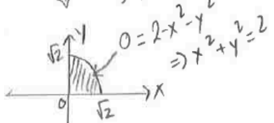
(b) Let E_2 be the solid region in the first octant that lies under the paraboloid $z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\,dV$ in cylindrical coordinates (you don't need to evaluate it).

Solution:

(b) Let E_2 be the solid region in the first octant that lies under the paraboloid $z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\,dV$ in cylindrical coordinates (you don't need to evaluate it).



projection onto xy-plane



$0 \le r \le \sqrt{2}$

$0 \le \theta \le \frac{\pi}{2}$

In cylindrical coordinates, $dV = r\,dz\,dr\,d\theta$

From the picture of E_2, we see that $0 \le z \le 2 - x^2 - y^2 = 2 - r^2$

To get the bounds for r and θ, we look at the projection of the solid E_2 onto the xy-plane. We see that

$0 \le r \le \sqrt{2}$ and $0 \le \theta \le \frac{\pi}{2}$.

So, $\iiint_{E_2} xz\,dV = \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_{z=0}^{z=2-r^2} (r\cos\theta) z\, r\,dz\,dr\,d\theta$ ($r\cos\theta = x$ in cylindrical coord.)

$$= \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_0^{2-r^2} z r^2 \cos\theta\,dz\,dr\,d\theta$$

3. Find the center of mass of the solid S bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$ if S has constant density 1 and total mass $\frac{\pi}{2}$. (Hint: $\overline{x}$ and $\overline{y}$ can be found by symmetry of the solid being considered).

Solution: Since the density is constantly 1, we just need to compute the average values of x, y and z inside this solid. Because the solid is rotationally symmetric about the z-axis, we get $\overline{x} = \overline{y} = 0$. Now we compute

$$\begin{aligned}
\overline{z} =& \frac{2}{\pi} \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{z}} zr dr d\theta dz \\
=& \frac{2}{\pi} \int_0^1 \int_0^{2\pi} z \frac{r^2}{2} \Big|_0^{\sqrt{z}} d\theta dz \\
=& \frac{2}{\pi} \int_0^1 \int_0^{2\pi} \frac{z^2}{2} d\theta dz \\
=& 2 \int_0^1 z^2 dz \\
=& \frac{2}{3},
\end{aligned}$$

so the center of mass is given by $(0, 0, \frac{2}{3})$.

4. Find the volume of the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

Solution: Let E denote the region given in the question. The volume of the solid is given by

$$\begin{aligned}
V &= \iiint_E dV \\
&= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 zr|_{z=r^2}^{z=1} \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^1 r - r^3 \, dr \, d\theta \\
&= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4} \Big|_0^1 \, d\theta \\
&= \int_0^{2\pi} 1/4 \, dz \\
&= \frac{\pi}{2}.
\end{aligned}$$

5. Use polar coordinates to show that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dA = \pi$$

and deduce that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.

Solution: We convert to polar coordinates, remembering that $dx\ dy$ becomes $r\ dr\ d\theta$. For the bounds, notice the original integral covers the entire plane. Thus we have

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\ dr\ d\theta$$

which now allows us to use u-substitution (which was impossible in the original integral). We take $u = r^2$, so that $du = 2r\ dr$. At the same time we may compute the integral over theta (which is 2π), so we have

$$\pi \int_0^{\infty} e^{-u} du = \pi$$

Now, since the original integrand is a separable function of x and y, i.e. it may be written as a product $e^{-x^2} e^{-y^2}$, and the region of integration is rectangular, our integrals are independent and we may write the original question as

$$\int_{-\infty}^{+\infty} e^{-x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-y^2} dy$$

If we think of y as a dummy variable, we notice that this is the integral we are trying to show equal to $\sqrt{\pi}$, times itself. This proves the desired result, since we have

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 = \pi$$

so

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

6. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on the ellipse that are nearest and farthest from the origin.

Solution: We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ (this corresponds to distance function from origin squared) subject to the two constraints

$g = x + y + 2z = 2$ and $h = x^2 + y^2 - z = 0$. Using the gradient equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

we obtain the system

$$\begin{cases} 2x = \lambda = 2\mu x \\ 2y = \lambda + 2\mu y \\ 2z = 2\lambda - \mu \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{cases}$$

Solving the equations, we obtain the points $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$. Then we have $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ (which is closest to the origin) and $f(-1, -1, 2) = 6$ (which is farthest from the origin).

7. Set up, but do not solve, the integral that gives the volume of the solid region bounded by the paraboloid $z = 3x^2 + 3y^2$ and the cone $z = 4 - \sqrt{x^2 + y^2}$.

Solution: The region of integration will be the interior of the projection of the curve of intersection of $z = 3x^2 + 3y^2$ with $z = 4 - \sqrt{x^2 + y^2}$. Setting the two equal to each other, we have

$$3x^2 + 3y^2 = 4 - \sqrt{x^2 + y^2}$$

and due to the appearence of sums of x^2 and y^2, we choose to convert to polar coordinates. This choice is reinforced by the rotational symmetry of our solid along z-axis. Setting $x = r\cos\theta$ and $y = r\sin\theta$, the equation above becomes

$$3r^2 = 4 - r$$

After rearranging as $3r^2 + r - 4 = 0$, we can factor it

$$(3r + 4)(r - 1) = 0$$

and the only nonnegative solution is $r = 1$. Then our integral should be expressible as an integral over $\theta \in [0, 2\pi]$ and $r \in [0, 1]$. We do top function (cone) minus bottom function (paraboloid), to get

$$\iint_R \left(4 - \sqrt{x^2 + y^2} - (3x^2 + 3y^2)\right) dxdy = \int_0^{2\pi} \int_0^1 (4 - r - 3r^2) r \; dr \; d\theta$$