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M20550 Calculus III Tutorial
Worksheet 7

1. Using spherical coordinates, compute the volume, $V(R)$ of a sphere of radius R.

> ***Solution:*** This is equivalent to just computing
>
> $$\iiint_{Sphere} dV$$
>
> (intuitively, we are summing up the volumes of infinitely many infinitesimally small boxes of volume "dV" inside the sphere.) Recall that the standard spherical coordinates are
>
> $$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$
>
> for $(\rho, \theta, \phi) \in [0, R] \times [0, 2\pi) \times (0, \pi)$ and the volume element of the sphere with respect to these coordinates is given by $dV = \rho^2 sin\phi d\theta d\phi d\rho$. So,
>
> $$\begin{aligned} V(R) = &\int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 sin\phi d\theta d\phi d\rho \\ =&2\pi \int_0^R \int_0^\pi \rho^2 sin\phi d\phi d\rho \\ =&4\pi \int_0^R \rho^2 d\rho \\ =&\frac{4}{3}\pi R^3 \end{aligned}$$

2. Now compute the surface area, $A(R)$, of a sphere of radius R. Hint: Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

And recall the common problem from single variable calculus where you have to find the volume of a water tank of height h by integrating the cross sectional area, $A(y)$, over the height.

$$Volume(Tank) = \int_0^h A(y)dy$$

We have a similar formula for the volume of the sphere;

$$V(R) = \int_0^R A(\rho)d\rho.$$

Solution: Let $A(\rho)$ be the surface area of the sphere of radius ρ, we wish to find $A(R)$. Observe

$$\int_0^R A(\rho)d\rho = V(R) = \frac{4}{3}\pi R^3$$

So by the fundamental theorem of calculus, we get

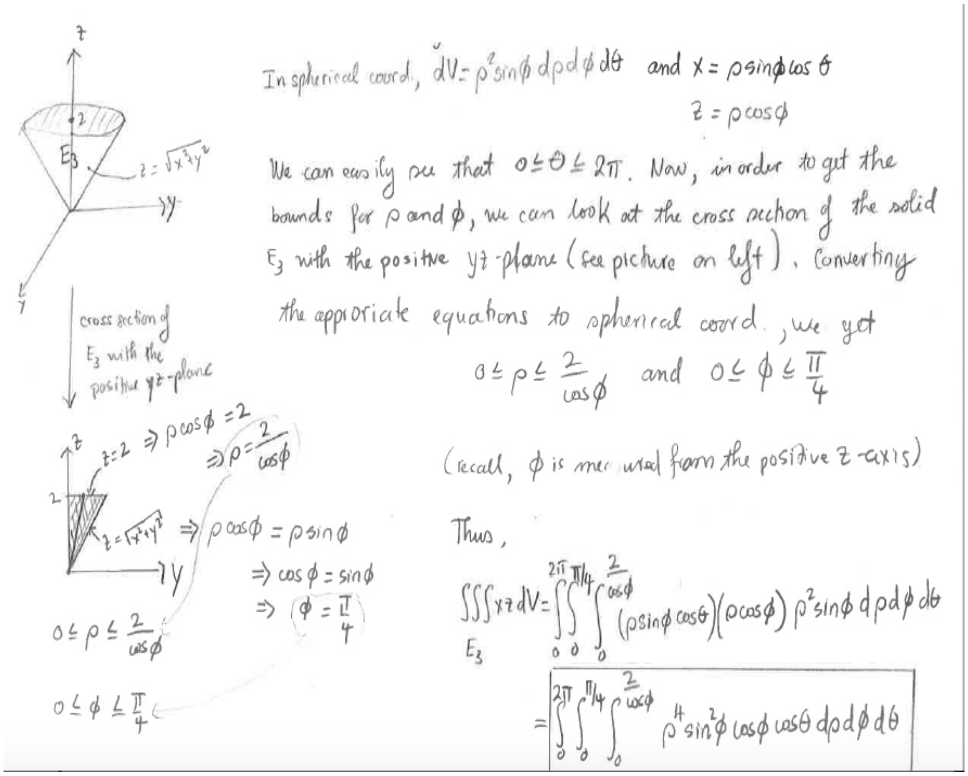
$$A(R) = \frac{d}{dR}\left[\int_0^R A(\rho)d\rho\right] = \frac{dV(R)}{dR} = 4\pi R^2.$$

Another way to solve this problem is to realize through geometric intuition or by reasoning similar to the argument above that

$$A(R) = \int_0^\pi \int_0^{2\pi} R^2 sin\phi d\theta d\phi.$$

3. Let E_3 be the solid region that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$. Write the triple integral $\iiint_{E_3} xz\, dV$ in spherical coordinates (you don't need to evaluate it).

Solution:



In spherical coord, $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and $x = \rho\sin\phi\cos\theta$, $z = \rho\cos\phi$

We can easily see that $0 \le \theta \le 2\pi$. Now, in order to get the bounds for ρ and ϕ, we can look at the cross section of the solid E_3 with the positive yz-plane (see picture on left). Converting the appropriate equations to spherical coord., we get

$$0 \le \rho \le \frac{2}{\cos\phi} \quad \text{and} \quad 0 \le \phi \le \frac{\pi}{4}$$

(recall, ϕ is measured from the positive z-axis)

Thus,

$$\iiint_{E_3} xz\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{2}{\cos\phi}} (\rho\sin\phi\cos\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{2}{\cos\phi}} \rho^4\sin^2\phi\cos\phi\cos\theta\,d\rho\,d\phi\,d\theta$$

4. Find the mass of the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ whose density is $\delta(x, y, z) = x^2 + y^2 + z^2$.

Solution: Let E be the solid in consideration. Now, to find the mass, we simply integrate the density function over the entire solid to get;

$$\begin{aligned}\int_1^2\int_0^{\pi}\int_0^{2\pi} \delta(\rho)\rho^2 sin\phi d\theta d\phi d\rho &= \int_1^2 A(\rho)\delta(\rho)d\rho \\ &= \int_1^2 4\pi\rho^2\rho^2 d\rho \\ &= 4\pi\frac{\rho^5}{5}\bigg|_1^2 \\ &= 4\pi(\frac{32}{5} - \frac{1}{5}) \\ &= \frac{124\pi}{5}.\end{aligned}$$

Note: The fact that the density only depended on ρ simplified our work here.

5. In this problem, we are going to calculate the same integral in two different ways by

changing coordinates. Compute the following integral;

$$\int_0^1 \int_0^1 x^3 y dx dy$$

first, by making the coordinate change $u = x^2, v = xy$, and then as you normally would. (Don't forget to multiply by the Jacobian!)

Solution:
We first compute the Jacobian;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ \frac{-v}{u^{\frac{3}{2}}} & \frac{1}{\sqrt{u}} \end{vmatrix} = \frac{1}{2u}$$

(note: u is always positive so we don't need to take the absolute value) now, we know by the change of variables formula that

$$\int_0^1 \int_0^1 x^3 y dx dy = \int_0^1 \int_0^{\sqrt{u}} uv \frac{1}{2u} dv du = \int_0^1 \frac{v^2}{4}\bigg|_{v=0}^{v=\sqrt{u}} du = \int_0^1 \frac{u}{4} du = \frac{1}{8}.$$

If we compute this integral in the usual way, we get;

$$\int_0^1 \int_0^1 x^3 y dx dy = \int_0^1 \frac{y}{4} dy = \frac{1}{8}.$$

6. Let R be the parallelogram enclosed by the lines $x + 3y = 0$, $x + 3y = 2$, $x + y = 1$, and $x + y = 4$. Evaluate the following integral by making appropriate change of variables

$$\iint_R \frac{x + 3y}{(x + y)^2} \, dA.$$

Solution: Observe the set of equations:

$$x + 3y = 0 \qquad\qquad x + 3y = 2$$
$$x + y = 1 \qquad\qquad x + y = 4$$

So, if we let

$$u = x + 3y \quad \text{ and } v = x + y,$$

then the transformation of R, denote S, is given by the region bounded by the lines

$$u = 0 \qquad u = 2$$
$$v = 1 \qquad v = 4$$

So, S is the region bounded by the rectangle $[0,2] \times [1,4]$ in the uv-plane.

Next, we need to compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

In order to compute these partials, we need to write x and y in terms of u and v. We have

$$x + 3y = u \quad (eq\ 1)$$
$$x + y = v \quad (eq\ 2)$$

$(eq\ 1) - (eq\ 2)$ is equivalent to $2y = u - v \implies y = \frac{1}{2}u - \frac{1}{2}v$. And $(eq\ 1) - 3(eq\ 2)$ gives $-2x = u - 3v \implies x = -\frac{1}{2}u + \frac{3}{2}v$. So,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\dfrac{1}{2} & \dfrac{3}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

Note that since $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial(u,v)}{\partial(x,y)}^{-1}$, we could have solved for the latter Jacobian instead and taken its reciprocal since it was a bit faster to compute in this case.

And so, we get

$$\begin{aligned} \iint_R \frac{x+3y}{(x+y)^2}\, dA &= \iint_S \frac{u}{v^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA \\ &= \int_1^4 \int_0^2 \frac{u}{v^2} \left| -\frac{1}{2} \right| du\, dv \\ &= \int_1^4 \frac{1}{4} u^2 v^{-2} \Big|_{u=0}^{u=2} dv \\ &= \int_1^4 v^{-2}\, dv \\ &= -\frac{1}{v}\Big|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}. \end{aligned}$$

7. Evaluate the line integral $\int_C (z-2xy)\, ds$ along the curve C given by $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$, $0 \le t \le \frac{\pi}{2}$.

Solution: $\int_C (z-2xy)\, ds$ is a line integral with respect to arc length (because of the ds at end). Since $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$, we get $x(t) = \sin t$, $y(t) = \cos t$, $z(t) = t$. So, $z - 2xy = t - 2\sin t \cos t$. And $\mathbf{r}'(t) = \langle \cos t, -\sin t, 1\rangle$. So,

$$ds = |\mathbf{r}'(t)|dt = \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt = \sqrt{\cos^2 t + (-\sin t)^2 + 1^2}\, dt = \sqrt{2}\, dt.$$

Thus, for $0 \le t \le \frac{\pi}{2}$,

$$\begin{aligned}\int_C (z-2xy)\, ds &= \int_0^{\pi/2} (t - 2\sin t\cos t)\sqrt{2}\, dt \\ &= \sqrt{2}\left[\frac{1}{2}t^2 - \sin^2 t\right]_0^{\pi/2} \\ &= \sqrt{2}\left[\frac{\pi^2}{8} - 1\right].\end{aligned}$$

8. Find $\int_C 2xy^3\, ds$ where C is the upper half of the circle $x^2 + y^2 = 4$.

Solution: First, let's parametrize the curve C. C is the upper half of the circle $x^2 + y^2 = 4$. So, we can let

$$x(t) = 2\cos t, \qquad y(t) = 2\sin t \qquad \text{for } 0 \le t \le \pi.$$

Then, $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$. Therefore,

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt = \sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 2\, dt.$$

Thus, for $0 \le t \le \pi$,

$$\begin{aligned}\int_C 2xy^3\, ds &= \int_0^{\pi} 2\,(2\cos t)\,(2\sin t)^3\; 2\, dt \\ &= \int_0^{\pi} 64\left(\sin^3 t\right)(\cos t)\; dt \\ &= 16\left[\sin^4 t\right]_0^{\pi} \\ &= 0.\end{aligned}$$