Name: Date: 04/08/2020

M20550 Calculus III Tutorial
Worksheet 9

1. Calculate the line integral $\int_C (y^2+x)\,dx+4xy\,dy$ where C is the arc of $x=y^2$ from $(1,1)$ to $(4,2)$.

Solution: First, we need to parametrize the curve C. Since C is a part of the curve $x=y^2$, we can let $y=t$; then we have $x=t^2$. Moreover, since the curve C is the part from $(1,1)$ to $(4,2)$, we get $1\le y\le 2$. So, we have $1\le t\le 2$. Thus, a parametrization of C is as follows:

$$x(t)=t^2, \qquad y(t)=t \qquad \text{for } 1\le t\le 2.$$

Now, $\int_C (y^2+x)\,dx+4xy\,dy$ is a line integral with respect to x and y because we see the dx and dy. Here,

$$dx=x'(t)\,dt=2t\,dt \quad \text{and} \quad dy=y'(t)\,dt=1\,dt.$$

So, for $1\le t\le 2$,

$$\begin{aligned}\int_C (y^2+x)\,dx+4xy\,dy &= \int_1^2 \left[\left(t^2+t^2\right)2t+4(t^2)(t)\right]dt\\ &= \int_1^2 8t^3\,dt\\ &= \left[2t^4\right]_1^2\\ &= 2^5-2=30.\end{aligned}$$

2. Evaluate the line integral $\int_C z^2\,dx+x\,dy+y\,dz$ where C is the line segment from $(1,0,0)$ to $(4,1,2)$.

Solution: First, we parametrize C, the line segment **from** $(1,0,0)$ **to** $(4,1,2)$. For $0\le t\le 1$, C can be written as the vector function

$$\mathbf{r}(t)=\langle 1,0,0\rangle+t\Big(\langle 4,1,2\rangle-\langle 1,0,0\rangle\Big)=\langle 1,0,0\rangle+t\,\langle 3,1,2\rangle.$$

So, $x(t)=1+3t$, $y(t)=t$, and $z(t)=2t$ for $0\le t\le 1$. Then,

$$dx=x'(t)\,dt=3\,dt, \quad dy=y'(t)\,dt=1\,dt, \quad dz=z'(t)\,dt=2\,dt.$$

Hence, for $0 \leq t \leq 1$,

$$\begin{aligned}\int_C z^2\,dx + x\,dy + y\,dz &= \int_0^1 \Big[(2t)^2(3) + (1+3t)(1) + t(2)\Big]\,dt \\ &= \int_0^1 \big[12t^2 + 5t + 1\big]\,dt \\ &= \left[4t^3 + \frac{5}{2}t^2 + t\right]_0^1 \\ &= \frac{15}{2}.\end{aligned}$$

3. Compute $\int_C x^2\, ds$ where C is the intersection of the surface $x^2+y^2+z^2=4$ and the plane $z=\sqrt{3}$.

Solution: The intersection of the sphere $x^2+y^2+z^2=4$ and the plane $z=\sqrt{3}$ is the circle

$$x^2+y^2+\left(\sqrt{3}\right)^2=4, \quad z=\sqrt{3}$$

$$\text{or simply} \quad x^2+y^2=1, \quad z=\sqrt{3}.$$

Thus, a parametrization of C could be

$$\mathbf{r}(t)=\left\langle \cos t, \sin t, \sqrt{3}\right\rangle \quad \text{for } 0\le t\le 2\pi.$$

Then, $\mathbf{r}'(t)=\langle -\sin t, \cos t, 0\rangle \implies |\mathbf{r}'(t)|=\sqrt{(-\sin t)^2+\cos^2 t}=1$.
So $ds=|\mathbf{r}'(t)|\,dt=1\,dt$. Finally, for $0\le t\le 2\pi$,

$$\begin{aligned}\int_C x^2\,ds &= \int_0^{2\pi}\left(\cos^2 t\right)\,dt\\ &= \int_0^{2\pi}\frac{1}{2}\left(1+\cos 2t\right)dt\\ &= \frac{1}{2}\left[t+\frac{1}{2}\sin(2t)\right]_0^{2\pi}\\ &= \pi.\end{aligned}$$

4. Determine whether or not the following vector fields are conservative:

(a) $\mathbf{F}=(3+2xy)\,\mathbf{i}+(x^2-3y^2)\,\mathbf{j}$

(b) $\mathbf{F}=\mathbf{i}+\sin z\;\mathbf{j}+y\cos z\;\mathbf{k}$

Solution: (a) Since $\mathbf{F}$ is a vector field on $\mathbb{R}^2$, we use the criterion $\dfrac{\partial P}{\partial y}\overset{?}{=}\dfrac{\partial Q}{\partial x}$ to see if $\mathbf{F}$ is conservative or not. We have $\mathbf{F}=\langle 3+2xy, x^2-3y^2\rangle$. So, $P=3+2xy$ and $Q=x^2-3y^2$ and

$$\frac{\partial P}{\partial y}=2x=\frac{\partial Q}{\partial x}.$$

Since $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$, $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^2$.

(b) Since $\mathbf{F}$ is a vector field on $\mathbb{R}^3$, we use the criterion curl $\mathbf{F} \stackrel{?}{=} \mathbf{0}$ to see if $\mathbf{F}$ is conservative or not. We have $\mathbf{F} = \langle 1, \sin z, y\cos z\rangle$. And

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & \sin z & y\cos z \end{vmatrix} = \langle \cos z - \cos z, 0, 0\rangle = \langle 0,0,0\rangle = \mathbf{0}.$$

Since curl $\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^3$.

5. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -2xy\,\mathbf{i} + 4y\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2$.

Solution: Since $x = t$, $y = t^2$, $z = 1$, we have

$$\mathbf{F}(\mathbf{r}(t)) = -2t^3\mathbf{i} + 4t^2\mathbf{j} + \mathbf{k} = \langle -2t^3, 4t^2, 1\rangle,$$

and

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} = \langle 1, 2t, 0\rangle$$

The line integral of $\mathbf{F}$ along C is

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\, dt \\ &= \int_0^2 \langle -2t^3, 4t^2, 1\rangle \cdot \langle 1, 2t, 0\rangle\, dt \\ &= \int_0^2 \left(-2t^3 + 8t^3\right)\, dt \\ &= \int_0^2 6t^3\, dt \\ &= \left.\frac{6t^4}{4}\right|_0^2 \\ &= \frac{3\cdot 2^4}{2} - 0 \\ &= 24. \end{aligned}$$

Remark: Note that $\mathbf{F}$ is not a conservative vector field, so we cannot apply the Fundamental Theorem of Line Integrals in this example. To see this note that

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2xy & 4y & 1 \end{vmatrix} = \langle 0, 0, 2x\rangle \neq \mathbf{0}.$$

6. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (y^2\cos(xy^2)+3x^2)\,\mathbf{i} + (2xy\cos(xy^2)+2y)\,\mathbf{j}$ is a conservative vector field and C is any curve from the point $(-1,0)$ to $(1,0)$.

Solution: Since we know $\mathbf{F}$ is a conservative vector field, $\mathbf{F} = \nabla f$ for some scalar function $f(x,y)$. So, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$. Then, by the fundamental theorem of line integral (FTLI), we have $\int_C \nabla f\cdot d\mathbf{r} = f(1,0) - f(-1,0)$. So, let's go about and find the potential function $f(x,y)$ of $\mathbf{F}$ first.

We know $\mathbf{F} = \nabla f$, so $\langle y^2\cos(xy^2)+3x^2, 2xy\cos(xy^2)+2y\rangle = \langle f_x, f_y\rangle$. Thus, we have

$$f_x = y^2\cos(xy^2) + 3x^2 \tag{1}$$
$$f_y = 2xy\cos(xy^2) + 2y \tag{2}$$

Using equation (1), we have $f = \int (y^2\cos(xy^2)+3x^2)\,dx = \sin(xy^2)+x^3+g(y)$. Now, we need to find $g(y)$ to complete f.

With $f = \sin(xy^2) + x^3 + g(y)$, we compute $f_y = 2xy\cos(xy^2) + g'(y)$. Then from equation (2) above, we must have

$$2xy\cos(xy^2) + g'(y) = 2xy\cos(xy^2) + 2y \implies g'(y) = 2y \implies g(y) = y^2 + C.$$

We only need *a* potential function to apply FTLI, so we can pick $C = 0$. So, a potential function $f(x,y)$ of the vector field $\mathbf{F}$ is

$$f(x,y) = \sin(xy^2) + x^3 + y^2.$$

Finally,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} &\overset{\text{FTLI}}{=} f(1,0) - f(-1,0)\\ &= (\sin 0 + 1^3 + 0^2) - (\sin 0 + (-1)^3 + 0^2)\\ &= 2.\end{aligned}$$

7. Use Green's Theorem to evaluate

$$\int_C \left(-\frac{y^3}{3} + \sin x\right)\,dx + \left(\frac{x^3}{3} + y\right)\,dy,$$

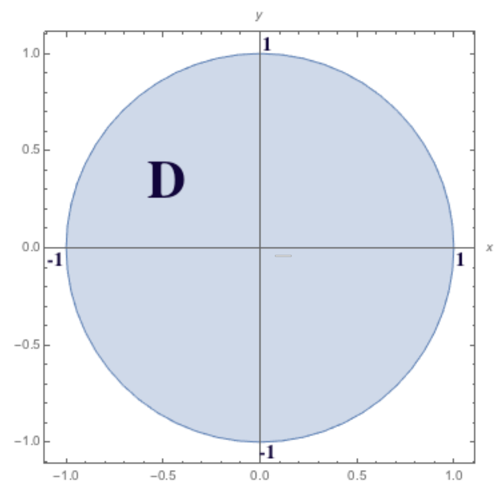
where C is the circle of radius 1 centered at $(0,0)$ oriented counterclockwise when viewed from above.

Solution: Let D be the region enclosed by the unit circle C in this problem. By Green's Theorem, we have

$$\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy = \iint_D x^2 - (-y^2)\, dA.$$

$\left(\text{Here, we have } P = -\frac{y^3}{3} + \sin x \text{ and } Q = \frac{x^3}{3} + y, \text{ so } \frac{\partial P}{\partial y} = -y^2 \text{ and } \frac{\partial Q}{\partial x} = x^2.\right)$

So, instead of computing the line integral $\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy$, we are going to compute the double integral $\iint_D x^2 + y^2\, dA$, where D is the unit disk as shown below.



Using polar coordinates,

$$\iint_D x^2 + y^2\, dA = \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = 2\pi \left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

Hence,

$$\int_C \left(-\frac{y^3}{3} + \sin x\right) dx + \left(\frac{x^3}{3} + y\right) dy = \frac{\pi}{2}.$$