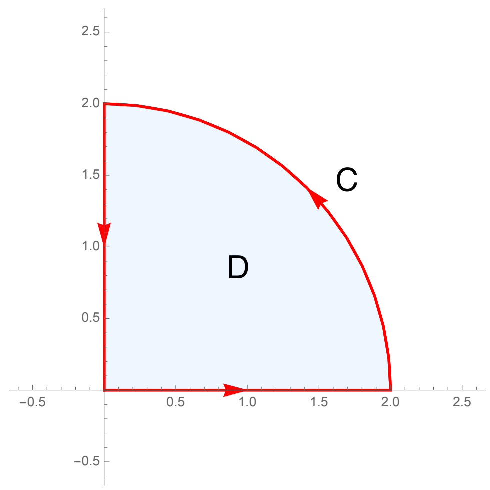
Name: Date: 4/16/2020

M20550 Calculus III Tutorial
Worksheet 10

1. A particle starts at the origin $(0,0)$, moves along the x-axis to $(2,0)$, then along the curve $y = \sqrt{4-x^2}$ to the point $(0,2)$, and then along the y-axis back to the origin. Find the work done on this particle by the force field $\mathbf{F}(x,y) = y^2\,\mathbf{i} + 2x(y+1)\,\mathbf{j}$.

Solution: First we note that the curve C (drawn below) is a positively oriented, piecewise-smooth, simple closed curve in the plane. Let D be the region bounded by C.



The components of the vector field, $P = y^2$ and $Q = 2x(y+1)$, have continuous partial derivatives on an open region containing D (namely, all of $\mathbb{R}^2$). We may apply Green's Theorem:

$$\int_C P\,dx + Q\,dy = \int\!\!\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

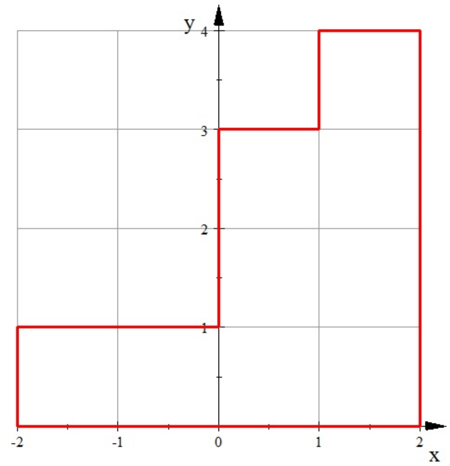
Note that we have $\frac{\partial Q}{\partial x} = 2(y+1) = 2y+2$ and $\frac{\partial P}{\partial y} = 2y$. Finally, we compute the work done on the particle by the force field.

$$\begin{aligned} W = \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C y^2\,dx + 2x(y+1)\,dy \\ &\overset{Green}{=} \int\!\!\int_D (2y+2-2y)\,dA \\ &= 2\int\!\!\int_D dA \end{aligned}$$

Note that this is just twice the area of the region D. We may compute this as a double integral using polar coordinates $\left(W = 2\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta\right)$ or by using the formula for the area of a circle. Thus,

$$W = 2(\text{Area of } D) = 2\left(\frac{\pi \cdot 2^2}{4}\right) = 2\pi.$$

2. Evaluate $\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$ where C is the curve below and C is oriented in the clockwise direction.



Solution: This problem uses Green's theorem. One main clue is the shape of the curve C (it has 8 pieces!). Let D be the region enclosed by the curve C. And since the orientation of C is clockwise, instead of counterclockwise, we have

$$\begin{aligned}
\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy &= -\iint_D \left[(3 + 5x^4y^4) - (5x^4y^4 - 2)\right] dA \\
&= -\iint_D 5\,dA \\
&= -5\iint_D 1\,dA \\
&= -5 \cdot \text{Area}(D) \\
&= -5 \cdot 9 \\
&= -45.
\end{aligned}$$

3. Compute div $\mathbf{F}$ and curl $\mathbf{F}$ for the following vector fields.
   (a) $\mathbf{F} = x^2y\mathbf{i} - (z^3 - 3x)\mathbf{j} + 4y^2\mathbf{k}$
   (b) $\mathbf{F} = (3x + 2z^2)\mathbf{i} + \frac{x^3y^2}{z}\mathbf{j} - (z - 7x)\mathbf{k}$

Solution: (a) div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-(z^3 - 3x) + \frac{\partial}{\partial z}(4y^2) = 2xy$.
For the curl, we compute

$$\begin{aligned} \operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -(z^3 - 3x) & 4y^2 \end{vmatrix} \\ &= (8y + 3z^2)\mathbf{i} + (3 - x^2)\mathbf{k} \end{aligned}$$

(b) div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(3x + 2z^2) + \frac{\partial}{\partial y}\left(\frac{x^3y^2}{z}\right) + \frac{\partial}{\partial z}(-(z - 7x)) = 2 + \frac{2x^3y}{z}$.
Again, we have

$$\begin{aligned} \operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x + 2z^2 & \frac{x^3y^2}{z} & -(z - 7x) \end{vmatrix} \\ &= \frac{x^3y^2}{z^2}\mathbf{i} + (4z - 7)\mathbf{j} + \frac{3x^2y^2}{z}\mathbf{k} \end{aligned}$$

4. (a) Compute div $\mathbf{F}$, where $\mathbf{F} = \langle e^y, zy, xy^2 \rangle$.
   (b) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2 \rangle$? Why?

Solution: (a) div $\mathbf{F} = \dfrac{\partial}{\partial x}(e^y) + \dfrac{\partial}{\partial y}(zy) + \dfrac{\partial}{\partial z}(xy^2) = 0 + z + 0 = z$
(b) For this problem, we need to remember the fact

$$\text{div curl } \mathbf{F} = 0 \quad \text{for any vector field } \mathbf{F}.$$

If there is a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2 \rangle$ then by the fact above, $\mathbf{G}$ would satisfy the rule

$$\operatorname{div}\operatorname{curl}\mathbf{G} = 0 \quad \text{or} \quad \operatorname{div}\left\langle xyz, -y^2z, yz^2 \right\rangle = 0.$$

But,

$$\operatorname{div}\left\langle xyz, -y^2z, yz^2 \right\rangle = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(yz^2) = yz - 2yz + 2yz = yz \neq 0.$$

Thus, there is no such $\mathbf{G}$.

5. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$$

is incompressible (i.e. has div $\mathbf{F}$= 0 everywhere).

Solution: We have

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x} f(y, z) + \frac{\partial}{\partial y} g(x, z) + \frac{\partial}{\partial z} h(x, y) \\ &= 0. \end{aligned}$$

Note that the partial derivatives are all 0 because, for example, $f(y, z)$ is not a function of x and therefore its partial derivative with respect to x is 0.

6. Fill in the sentences below by circling one option:

We can take the gradient of a (vector field/scalar function), and the output is a (vector field/scalar function).

We can take the divergence of a (vector field/scalar function) and the output is a (vector field/scalar function).

We can take the curl of a (vector field/scalar function), and the output is a (vector field/scalar function).

Solution: We can take the gradient of a **scalar function**, and the output is a **vector field**.

We can take the divergence of a **vector field** and the output is a **scalar function**.

We can take the curl of a **vector field**, and the output is a **vector field**.

7. Which of the following combinations of grad, div, and curl make sense?

$$\begin{array}{ccc} \nabla(\nabla(f)) & div(\nabla(f)) & curl(\nabla(f)) \\ \nabla(div(\mathbf{F})) & div(div(\mathbf{F})) & curl(div(\mathbf{F})) \\ \nabla(curl(\mathbf{F})) & div(curl(\mathbf{F})) & curl(curl(\mathbf{F})) \end{array}$$

Solution: The combinations that make sense here are $div(\nabla(f))$, $curl(\nabla(f))$, $\nabla(div(\mathbf{F}))$, $div(curl(\mathbf{F}))$, and $curl(curl(\mathbf{F}))$. In all of the other combinations, the output of the inner function is not a valid input for the outer function. For example, $\nabla(f)$ is a vector field, so we cannot evaluate $\nabla(\nabla(f))$ since ∇ takes scalar functions as inputs.