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Worksheet 11

1. Compute the surface integral $\iint_S (x+y+z)\, dS$, where S is a surface given by $\mathbf{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$ and $0 \le u \le 2,\ 0 \le v \le 1$.

> ***Solution:*** First, we know
>
> $$\iint_S (x+y+z)\, dS = \iint_D \Big[(u+v)+(u-v)+(1+2u+v)\Big] |\mathbf{r}_u \times \mathbf{r}_v|\, dA,$$
>
> where D is the domain of the parameters $u,\ v$ given by $0 \le u \le 2,\ 0 \le v \le 1$.
> We have $\mathbf{r}_u = \langle 1,1,2\rangle$ and $\mathbf{r}_v = \langle 1,-1,1\rangle$. Then,
> $\mathbf{r}_u \times \mathbf{r}_v = \langle 1,1,2\rangle \times \langle 1,-1,1\rangle = \langle 3,1,-2\rangle$. So,
>
> $$|\mathbf{r}_u \times \mathbf{r}_v| = |\langle 3,1,-2\rangle| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}.$$
>
> Thus,
>
> $$\begin{aligned}\iint_S (x+y+z)\, dS &= \int_0^1 \int_0^2 (4u+v+1)\sqrt{14}\, du\, dv \\ &= 11\sqrt{14}.\end{aligned}$$

2. Let S be the portion of the graph $z = 4 - 2x^2 - 3y^2$ that lies over the region in the xy-plane bounded by $x=0,\ y=0$, and $x+y=1$. Write the integral that computes $\iint_S \left(x^2+y^2+z\right) dS$.

> ***Solution:*** First, we need a parametrization of the surface S. Since S is a surface given by the equation $z = 4-2x^2-3y^2$, we can choose x and y to be the parameters. So,
>
> $$\mathbf{r}(x,y) = \left\langle x, y, 4-2x^2-3y^2\right\rangle,$$
>
> and the domain D of the parameters $x,\ y$ is given by the region in the xy-plane bounded by $x=0,\ y=0$, and $x+y=1$ (see picture below)

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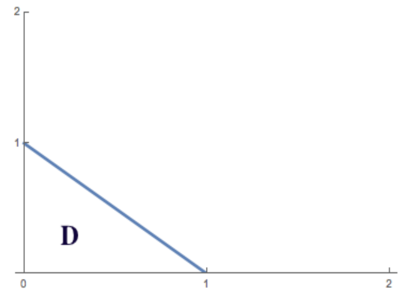

Now, $\mathbf{r}_x = \langle 1, 0, -4x \rangle$ and $\mathbf{r}_y = \langle 0, 1, -6y \rangle$. So, $\mathbf{r}_x \times \mathbf{r}_y = \langle 4x, 6y, 1 \rangle$ and $|\mathbf{r}_x \times \mathbf{r}_y| = |\langle 4x, 6y, 1 \rangle| = \sqrt{16x^2 + 36y^2 + 1}$. Thus,

$$\iint_S \left(x^2 + y^2 + z\right)\, dS = \iint_D x^2 + y^2 + (4 - 2x^2 - 3y^2)|\mathbf{r}_x \times \mathbf{r}_y|\, dA$$
$$= \int_0^1 \int_0^{-x+1} (4 - x^2 - 2y^2)\sqrt{16x^2 + 36y^2 + 1}\; dy\, dx.$$

3. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is a surface given by

$$x = 2u, \quad y = 2v, \quad z = 5 - u^2 - v^2,$$

where $u^2 + v^2 \leq 1$. S has downward orientation.

Solution: We have $\mathbf{r}(u, v) = \langle 2u, 2v, 5 - u^2 - v^2 \rangle$, so $\mathbf{r}_u = \langle 2, 0, -2u \rangle$ and $\mathbf{r}_v = \langle 0, 2, -2v \rangle$ and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2, 0, -2u \rangle \times \langle 0, 2, -2v \rangle = \langle 4u, 4v, 4 \rangle.$$

Note that $\mathbf{r}_u \times \mathbf{r}_v = \langle 4u, 4v, 4 \rangle$ gives unit normal vectors pointing upward (z-component is positive). But, S has downward orientation so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\iint_{u^2+v^2 \leq 1} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

Now, $\mathbf{F}\big(\mathbf{r}(u, v)\big) = \langle 2v, -2u, 5 - u^2 - v^2 \rangle$. So

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \left\langle 2v, -2u, 5 - u^2 - v^2 \right\rangle \cdot \langle 4u, 4v, 4 \rangle = 20 - 4u^2 - 4v^2.$$

Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint\limits_{u^2+v^2\le 1} (20-4u^2-4v^2)\,dA.$$
$$\overset{\text{polar}}{=} -\int_0^{2\pi}\int_0^1 (20-4r^2)r\,dr\,d\theta$$
$$= -18\pi.$$

4. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

Solution: We want to compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin.

Note that this is not a closed surface (it has no top nor bottom), otherwise, we would use Divergence Theorem. This flux integral doesn't seem to be difficult to compute directly. First, we parametrize S: let $x = 2\cos u$, $y = 2\sin u$, $z = v$. Then

$$\mathbf{r}(u,v) = \langle 2\cos u, 2\sin u, v\rangle, \quad \text{domain } D \text{ is } \ 0\le u\le 2\pi,\ 0\le v\le 2.$$

Then, $\mathbf{r}_u = \langle -2\sin u, 2\cos u, 0\rangle$ and $\mathbf{r}_v = \langle 0,0,1\rangle$. So,

$$\mathbf{r}_u\times\mathbf{r}_v = \langle -2\sin u, 2\cos u, 0\rangle \times \langle 0,0,1\rangle = \langle 2\cos u, 2\sin u, 0\rangle.$$

Now, let's check our orientation. Let's take the point where $u = \pi/2$ and $v = 1$, ie $(x,y,z) = (0,2,1)$. At the point $(0,2,1)$, the unit normal vector points in the direction of the vector $(\mathbf{r}_u\times\mathbf{r}_v)(\pi/2,1) = \langle 0,2,0\rangle$. This means the unit normal vector is pointing away from the origin. So, our parametrization of S gives the correct orientation for S. Moving on!

Now, $\mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot(\mathbf{r}_u\times\mathbf{r}_v) = \langle 2\cos u, 2\sin u, v\rangle\cdot\langle 2\cos u, 2\sin u, 0\rangle = 4.$

Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\ dA$$
$$= \iint_D 4\,dA.$$
$$= \int_0^{2\pi}\int_0^2 4\,dv\,du$$
$$= 16\pi.$$

5. Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, z, 1\rangle$ across the hemi-sphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ with orientation away from the origin.

Solution: If we do this problem from scratch, we need to start by parametrizing the hemi-sphere:

$$x(\phi, \theta) = 2\sin\phi\cos\theta, \qquad y(\phi, \theta) = 2\sin\phi\sin\theta, \qquad z(\phi, \theta) = 2\cos\phi,$$

where $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$. Then $\mathbf{r}(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$, where $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$. And we get

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi\rangle$$

We now want to check the orientation of the surface. Let $\phi = \pi/4$ and $\theta = \pi/2$, then at the point $(0, \sqrt{2}, \sqrt{2})$, we get the vector $\mathbf{r}_\phi \times \mathbf{r}_\theta(\pi/4, \pi/2) = \langle 0, 2, 2\rangle$ points away from the origin. Thus, our parametrization gives the correct orientation of the surface.

Then, we have the flux of $\mathbf{F}$ across the given hemi-sphere H can be compute using the formula

$$\iint_H \mathbf{F}\cdot d\mathbf{S} = \iint_{\substack{0\leq\phi\leq\pi/2\\0\leq\theta\leq2\pi}} \mathbf{F}\big(\mathbf{r}(\phi,\theta)\big)\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA.$$

$\mathbf{F}\big(\mathbf{r}(\phi,\theta)\big) = \langle 0, 2\cos\phi, 1\rangle$ and

$$\mathbf{F}\big(\mathbf{r}(\phi,\theta)\big)\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = 8\sin^2\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi$$

Thus,

$$\begin{aligned}
\iint_H \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} \big(8\sin^2\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi\big)\, d\phi\, d\theta\\
&= \int_0^{2\pi}\left(\frac{8}{3}\sin^3\phi\Big|_0^{\pi/2}\sin\theta + 2\sin^2\phi\Big|_0^{\pi/2}\right) d\theta\\
&= \int_0^{2\pi}\left(\frac{8}{3}\sin\theta + 2\right) d\theta\\
&= -\frac{8}{3}\cos\theta\Big|_0^{2\pi} + 2\cdot 2\pi\\
&= 4\pi
\end{aligned}$$

Another Solution: If you already know that for a sphere of radius 2 with orientation away from the origin, its unit normal vector is given by $\mathbf{n} = \left\langle \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z\right\rangle$ and

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$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$, then we could use the definition of the flux integral to compute $\iint_H \mathbf{F}\cdot d\mathbf{S}$ as follows:

$$\begin{aligned}
\iint_H \mathbf{F}\cdot d\mathbf{S} &= \iint_H \mathbf{F}\cdot\mathbf{n}\,dS \\
&= \iint_D \langle 0, z, 1\rangle \cdot \left\langle \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z \right\rangle\, dS \\
&= \iint_H \left(\frac{1}{2}yz + \frac{1}{2}z\right)\, dS \\
&= \iint\limits_{\substack{0\le\phi\le\pi/2 \\ 0\le\theta\le 2\pi}} (2\sin\phi\cos\phi\sin\theta + \cos\phi)\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA \\
&= \int_0^{2\pi}\int_0^{\pi} (2\sin\phi\cos\phi\sin\theta + \cos\phi)\, 4\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \left(8\sin^2\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi\right)\, d\phi\, d\theta \\
&= 4\pi
\end{aligned}$$