# 0. Introduction to stable homotopy category

Note Title 8/27/2014

"Prerequisites"

Category theory

- categories + functors
- adjoint functors
- limits + colimits

- Triangulated category

Homotopy theory $\wedge, \vee$ $\pi_*$, LES of $\pi_*$

- $\Omega, \Sigma$ $[\Sigma X, Y]_* \simeq [X, \Omega Y]_*$ $\pi_n(\Omega X) \cong \pi_{n+1}(X)$
- CW approx + Whitehead thm $\tilde{H}^n(X) = \tilde{H}^{n+1}(\Sigma X)$
- $X \xrightarrow{f} Y \to C(f)$ cofiber sequence

$$[C(f), Z]_* \to [Y, Z]_* \to [X, Z]_*$$

$X \to Y$ cofibration $\Rightarrow C(f) \simeq Y/X$

(e.g. sub CW cx.)

- $F(f) \to X \xrightarrow{f} Y$, $[Z, -]_*$ yields exact seq.

$X \xrightarrow{f} Y$ fibration

e.g. locally trivial bundle / paracompact base

$F(f) \simeq f^{-1}(*)$

- EM spaces

$$[X, K(\pi, n)] = H^n(X)$$

# Stable homotopy category

## Homotopy Category

$Top/\simeq$

Objects: top'l spaces

Morphisms: $[X,Y] = Map(X,Y)/\simeq$

$$Ho(Top) = Top[w.h.e.^{-1}]$$

$$= CW/\simeq$$

CW approx: every space is weakly equivalent to CW complex

whitehead thm: w.e. between CW cx's $\Rightarrow$ h.e.

$Top_*$ = cat of pointed top'l spaces

$$Ho(Top_*) = Top_*[w.h.e^{-1}]$$

$$= CW_*/\simeq$$

pointed homotopy

Stable htpy Cat = $Ho(Sp)$   "$Sp$ = category of spectra"

SHC

2 perspectives

(1) "Stabilize" $Ho(Top_*)$

$$X \rightsquigarrow \Sigma X$$

$$SHC = \text{"} Ho(Top_*)[\Sigma^{-1}] \text{"}$$

Spanier-Whitehead category

objects $(X,n)$   "$\Sigma^{-n}X$"   $X \in CW_*$

$(X,n) \simeq (\Sigma X, n+1)$

morphisms

$$[(X,n),(Y,m)]^s = \varinjlim [\Sigma^{k-n}X, \Sigma^{k-m}Y]_*$$

Motivation: Stable homotopy groups

$$\pi_k^s(X) := \varinjlim_n \pi_{k+n}(\Sigma^n X)$$

suspension excision

$$= [(S^0,-k),(X,0)]^s$$

## (2) "Category of cohomology theories"

A (reduced) cohomology theory is a collection of functors $\{\tilde{E}^k\}_{k \in \mathbb{Z}}$

$$\tilde{E}^k : Top_* \longrightarrow Ab$$

and natural isomorphisms

$$\sigma : \tilde{E}^k(X) \xrightarrow{\cong} \tilde{E}^{k+1}(\Sigma X)$$

[unreduced $E^k(X) := \tilde{E}^k(X_+)$]

- homotopy invariant

$$f : X \xrightarrow[\text{w.h.e.}]{} Y \Rightarrow f^* : \tilde{E}^k(Y) \xrightarrow{\cong} \tilde{E}^k(X)$$

- excision $X \xrightarrow{f} Y \longrightarrow C(f)$

$$\Rightarrow \tilde{E}^k(C(f)) \longrightarrow \tilde{E}^k(Y) \longrightarrow \tilde{E}^k(X) \quad \text{exact}$$

- Wedge axiom

$$\tilde{E}^k(\vee X_i) \xrightarrow{\cong} \prod \tilde{E}^k(X_i)$$

**Stable natural transformation:**

$$f_k : \tilde{E}^k(-) \longrightarrow \tilde{F}^k(-)$$

compatible w/ suspension isos.

$$SHC = \text{"Coh Theories}/SNT\text{"}$$

Motivation: Eilenberg-Steenrod

Suppose $E$ is a coh thy

w/ $E^k(pt) = \begin{cases} \pi, & k=0 \\ 0, & o/w \end{cases}$

$$\Longrightarrow E^k(X) \cong H^k(X;\pi)$$

There are many interesting coh theories where this does not hold...

$\begin{cases} K\text{-thy} \\ \text{cobordism} \end{cases}$

How are these related? $\qquad E = $ coh thy.

$$\begin{tikzcd} Ho(Top_*) \arrow[r, "\tilde{E}^*"] \arrow[dr] & Gr(Ab) \\ & Ho(Top_*)[\Sigma^{-1}] \arrow[u, dashed, "\tilde{E}^*"'] \end{tikzcd}$$

$$\tilde{E}^k((X,n)) := \tilde{E}^{k+n}(X)$$

Warning: this is incorrect see next lecture

$$Ho(Top_*)[\Sigma^{-1}] \longrightarrow \text{Coh. thy's}$$

$$(X,n) \longmapsto \tilde{X}^k(Z) = [(Z,0),(X,n-k)]$$

$$\tilde{X}^0(S^k) = 0 \quad \text{for } k < n$$

Is this an iso??

i.e. is every coh theory represented in this way?

No. $\quad$ simplest example $\prod_i H^{k-i}(-)$

$$Ho(Top)[\Sigma^{-1}] \longrightarrow SHC \longrightarrow Coh.$$

not full and faithful

not an equiv of cats