

# 0. Introduction to stable homotopy category

Note Title

8/27/2014

## "Prerequisites"

### Category theory

- categories & functors
- adjoint functors
- limits & colimits
- Triangulated category

### Homotopy theory

- $\Omega, \Sigma$ :  $[\Sigma X, Y]_* \cong [X, \Omega Y]_*$   $\pi_n(\Omega X) \cong \pi_{n+1}(X)$
- CW approx & Whitehead theorem  $\pi_n(\Sigma X) \cong \pi_{n-1}(X)$
- $X \xrightarrow{f} Y \rightarrow C(f)$  cofiber sequence  $\tilde{H}^k(X) \cong \tilde{H}^{k+1}(C(f))$

$$[C(f), Z]_* \rightarrow [Y, Z]_* \rightarrow [X, Z]_*$$

$$X \rightarrow Y \text{ cofiber } \Rightarrow C(f) \cong Y/X$$

(e.g. sub CW com.)

- $F(f) \rightarrow X \xrightarrow{f} Y$ ,  $[Z, -]_*$  holds exact seq  
 $X \xrightarrow{f} Y$  fibration  
 ex. locally trivial bundle / principal bundle  
 $F(f) \cong f^{-1}(*)$

- EM spaces

$$[X, k(n, n)] = H^n(X)$$

# Stable homotopy category

## Homotopy Category

$$\text{Top} / \simeq$$

Objects: top'l spaces

$$\text{Morphisms: } [X, Y] = \text{Map}(X, Y) / \simeq$$

$$H_0(\text{Top}) = \text{Top} [\text{w.h.e.}^{-1}]$$

$$= \text{CW} / \simeq$$

CW approx  
 any space is weakly equivalent to CW complex  
Whitehead theorem  
 w.e. between CW cx's  $\Rightarrow$  h.e.

$\text{Top}_*$  = cat of pointed top'l spaces

$$H_0(\text{Top}_*) = \text{Top}_* [\text{w.h.e.}^{-1}]$$

$$= \text{CW}_* / \simeq$$

$\uparrow$  pointed homotopy

## Stable htpy Cat SHC

$$= H_0(\text{Sp})$$

"Sp = category of spectra"

2 perspectives

(1) "Stabilize"  $H_0(\text{Top}_*)$

$$X \rightsquigarrow \Sigma X$$

$$\text{SHC} = "H_0(\text{Top}_*)[\Sigma^{-1}]"$$

## Spectrum-Model category

objects  $(X, n)$  " $\Sigma^{-n} X$ "  $X \in \text{CW}_*$

$$(X, n) \simeq (\Sigma X, n+1) \xrightarrow{\text{morphisms}} [(X, n), (Y, m)]^s = \varinjlim [\Sigma^{k-n} X, \Sigma^{k-m} Y]_*$$

## Motivation

Stable homotopy groups

$$\pi_k^s(X) := \varinjlim_n \pi_{k+n}(\Sigma^n X)$$

$\uparrow$  stability excision

$$= [(S^0, -k), (X, 0)]^s$$

## (2) "Category of cohomology theories"

(reduced)  
A cohomology theory is a collection of functors  $\{\tilde{E}^k\}_{k \in \mathbb{Z}}$

$$\tilde{E}^k: \text{Top}_* \rightarrow \text{Ab}$$

and natural isomorphisms

$$\sigma: \tilde{E}^k(X) \xrightarrow{\cong} \tilde{E}^{k+1}(SX)$$

$$\left[ \begin{array}{l} \text{unreduced} \\ E^k(X) := \tilde{E}^k(X_+) \end{array} \right]$$

• homotopy invariant

$$f: X \rightarrow Y \quad \Rightarrow \quad f_*: \tilde{E}^k(Y) \xrightarrow{\cong} \tilde{E}^k(X)$$

w.h.e.

• exact

$$X \xrightarrow{f} Y \rightarrow C(f)$$

$$\Rightarrow \tilde{E}^k(C(f)) \rightarrow \tilde{E}^k(Y) \rightarrow \tilde{E}^k(X) \quad \text{exact}$$

• Wedge axiom

$$\tilde{E}^k(\bigvee X_i) \xrightarrow{\cong} \prod \tilde{E}^k(X_i)$$

Stable natural transformations:

$$f_k: \tilde{E}^k(-) \rightarrow \tilde{F}^k(-)$$

compatible w/ suspension isos.

SHT = "Coh Theories / SNT"

Motivation:

Eilenberg-Steenrod

Suppose  $E$  is a cohomology theory

$$w/ \quad E^k(pt) = \begin{cases} \pi, & k=0 \\ 0, & k \neq 0 \end{cases}$$

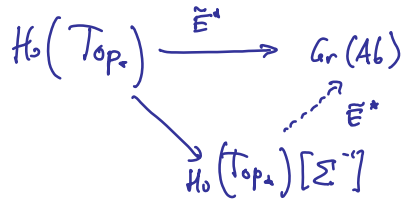
$$\Rightarrow E^k(X) \cong H^k(X; \pi)$$

There are many interesting cohomology theories other than this does not hold...

$\left\{ \begin{array}{l} K\text{-thy} \\ \text{cobordism} \end{array} \right.$

How are these related?

$E = \text{coh thy.}$



$$\tilde{E}^k((X, n)) := \tilde{E}^{k+n}(X)$$

Warning:  
 this is incorrect  
 see next lecture

$H_0(\text{Top}_*)[\Sigma^{-1}] \longrightarrow \text{Coh. thys}$

$$(X, n) \longmapsto \tilde{X}^k(\mathbb{Z}) = [(\mathbb{Z}, 0), (X, n-k)]$$

Is this an iso??

$$\tilde{X}^0(S^k) = 0 \text{ for } k < n$$

i.e. is every coh theory represented in this way?

No. stupid example  $\prod_i H^{k-i}(-)$

