11 - Cohomology operations

Wednesday, October 29, 2014 7:04 AM

$E \in Sp$, an Ω-Spectrum

Unstable Coh operations: $E^*: Top \to Gr(Set)$ $E^n(X) = [X, E_n]$

$$\underset{Set}{Nat}\left(E^n(-), E^m(-)\right) = [E_n, E_m] = E^m(E_n)$$

Note: these are NOT nec. gp homs

$$\text{''} \ \ Sets_* \ \ \text{''} \qquad = \tilde{E}^m(E_n)$$

Examples $E = H\mathbb{Q}$ $H\mathbb{Q}_n = K(\mathbb{Q}, n)$

$$H^*(K(\mathbb{Q},n);\mathbb{Q}) = \begin{cases} \Lambda_{\mathbb{Q}}[\iota_n], & n \text{ odd} \\ \mathbb{Q}[\iota_n], & n \text{ even} \end{cases} \quad \left\{ \begin{array}{l} \text{Related} \\ \text{to} \\ \pi_* S^n \otimes \mathbb{Q} \text{ computation} \end{array} \right\}$$

c.f. $n=1$, S^1; $n=2$, $\mathbb{C}P^\infty$

$$\lambda \iota_n \longleftrightarrow \cdot\lambda : H^n \longrightarrow H^n$$

$$\iota_n^i \longleftrightarrow (-)^i : H^n \longrightarrow H^{ni}$$

$$\underbrace{x \longmapsto x^i}_{\text{Not a hom.}}$$

$E = H\mathbb{F}_p$

$p=2$ $Sq^i : H\mathbb{F}_2^n(-) \longrightarrow H\mathbb{F}_2^{n+i}(-)$ homomorphisms

- $Sq^0 = Id$
- $Sq^i(xy) = \sum_{i_1+i_2} Sq^{i_1}(x)\, Sq^{i_2}(y)$
- if $|x| = i$, $Sq^i(x) = x^i$; $Sq^{2i}(x) = 0$

$\Big\}$ characterize these.

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

$p > 2$ $\quad P^i : H\mathbb{F}_p^{\,n}(-) \longrightarrow H\mathbb{F}_p^{\,n+2i(p-1)}(-)$

- $P^0 = Id$
- $P^i(xy) = \sum_{i_1+i_2=i} P^{i_1}(x)\, P^{i_2}(y)$
- if $|x| = 2i$, $P^i(x) = x^p$ $\quad$ If $2i > |x| \Rightarrow P^i(x) = 0$

$\beta : H\mathbb{F}_p^{\,n} \longrightarrow H\mathbb{F}_p^{\,n+1}$

(connecting hom $\quad 0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$)

[$p=2 \Rightarrow \beta = Sq^1$]

(Adem relations)

$p=2$ $\quad \mathcal{A} =$ algebra of all operations $= \mathbb{F}_2\langle Sq^i \rangle_{i \geq 1} / \text{Adem relations}$

"Steenrod algebra"

Adem relations $\Longrightarrow$ every elt of $\mathcal{A}$ is uniquely expressible as $\sum Sq^I$ $\quad Sq^I = Sq^{i_1} \cdots Sq^{i_s}$, $i_s \geq 2 i_{s+1}$

$X \in Top \Rightarrow H\mathbb{F}_2^*(X)$ is an "unstable $\mathcal{A}$-algebra"

Serre $\quad H^*(K(\mathbb{F}_2, n))$ is the free unstable $\mathcal{A}$-algebra on a generator in degree n

$$= \mathbb{F}_2\left[Sq^I(\iota_n) \;\middle|\; \begin{array}{l} I = (i_1, \ldots, i_s) \\ i_j \geq 2 i_{j+1} \\ i_j < i_{j+1} + \cdots + i_s + n \end{array} \right]$$

K operations:

$\lambda^i : K^0 \longrightarrow K^0$

$V \longmapsto \Lambda^i V$

Not homos

$\lambda^i(V \oplus V') = \sum_{i_1+i_2=i} \lambda^{i_1}(V)\, \lambda^{i_2}(V')$

e.g. $\quad V = L \qquad \lambda^0 L = 1$

$\lambda^1 L = L$

...

$\Rightarrow \lambda^i(L_1 + \cdots + L_n) = e_i(L_1, \ldots, L_n)$

$\lambda^{>1} L = 0$

$KU^0(\underline{KU}_0) = KU^0(BU \times \mathbb{Z})$

$KU^0(BU(n)) \hookrightarrow KU^0((\mathbb{C}P^\infty)^{\times n})^{\Sigma_n}$

$KU^0(\mathbb{C}P^\infty) = \mathbb{Z}[[x]]$

$x = [\xi] - 1$

"collapsing AHSS"

$$\cong$$

$$\left(\mathbb{Z}[\xi_1, \dots, \xi_n]^\wedge_{(\xi_1, \dots, \xi_n)}\right)^{\Sigma_n}$$

$= \mathbb{Z}[\xi]^\wedge_{(x)}$

$$\cong$$

$$\mathbb{Z}[e_1(\xi_i), \dots, e_n(\xi_i)]^\wedge \quad \Sigma_n$$

$$\parallel$$

$$\mathbb{Z}(\lambda^1(\xi_1 + \dots + \xi_n), \dots, \lambda^n(\xi_1 + \dots + \xi_n)]^\wedge$$

So $K^0(BU) \cong \mathbb{Z}(\lambda^1, \lambda^2, \dots]^\wedge$

$\cap$

$K^0(BU \times \mathbb{Z})$

$\exists!$ operation $\psi^i : K^0 \to K^0$

$$\psi^i(L_1 \oplus \dots \oplus L_n) = L_1^i \oplus \dots \oplus L_n^i$$

$\underline{\text{idea}}$: Write $x_1^i + \dots + x_n^i = s_i(e_1, \dots, e_n)$

$$\Rightarrow \psi^i(V) = s_i(\lambda^1(V), \dots, \lambda^n(V))$$

ψ act via ring maps

$\psi^{-1}(V) = \bar{V}$

$\psi^{-i} = \psi^{-1}\psi^i$

$\psi^i\psi^j = \psi^j\psi^i$

$\psi^p(x) \equiv x^p \mod p$

on $\tilde{K}^0(S^2)$ have $\psi^k(x) = kx$

$\Longrightarrow$ on $\tilde{K}^0(S^{2i})$ have $\psi^k(x) = k^i x$

$\cong$

$\mathbb{Z}$

$\underline{\text{Exercise}}$: check these.

Stable operations

Stable operations

$$E^*: Sp \longrightarrow Gr(Ab)$$

$$\mathrm{Nat}_{Gr(Ab)}\left(E^*(-), E^{*+i}(-)\right) = \left(\text{Coh. stable operations}\right)$$

$$\| \text{ Yoneda}$$

$$[E, \Sigma^i E]$$

$$\|$$

$$E^i E$$

$$E = \varinjlim_i \Sigma^{-i} \Sigma^\infty E_i$$

$$\Longrightarrow \quad 0 \to \lim{}^1 \to E^* E \to \varprojlim \tilde{E}^{*+i} E_i \to 0$$

$$\tilde{E}^n(E_i) \xrightarrow[\sigma]{} \tilde{E}^{n-1}(E_{i-1})$$

Given $\alpha_i : \tilde{E}^i(-) \to \tilde{E}^{i+n}(-)$

get $\tilde{E}^{i-1}(X) \cong \tilde{E}^i(\Sigma X) \xrightarrow[\alpha_i]{} \tilde{E}^i(X) \cong \tilde{E}^{i-1}(\Sigma X)$

$$0 \to \lim{}^1 \to \begin{matrix}\text{Stable} \\ \text{nat trans} \\ \text{of deg } k\end{matrix} \to \begin{matrix}\alpha_i : E^i(-) \to E^{i+n}(-) \\ \text{comp w/ susp.}\end{matrix} \to 0$$

E^*E is an algebra under composition.

If E is a commutative ring spectrum, get

$$E^*E \longrightarrow E^*(E \wedge E)$$ Künneth map - may be iso.

$$\uparrow \cong$$

$$E^*E \otimes_{E_*} E^*E$$

$\Rightarrow E^*E$ is a cocommutative coalgebra

e.g. $H\mathbb{F}_p^* H\mathbb{F}_p = \mathcal{A}$ $\qquad$ Sq^k is a stable operation ~ $\psi(Sq^k) = \dots$

U dies after a suspension

Stable Adams operations $\quad \psi^k : \tilde{K}^0 \to \tilde{K}^0$

$\mathrm{coprod}(\psi^k) = \psi^k \otimes \psi^k$

does it extend to a stable operation

e.g. $\quad \psi^k : BU \times \mathbb{Z} \to BU \times \mathbb{Z}$

$\qquad\qquad \Omega^\infty K \qquad \Omega^\infty K$

$\exists$ then a map $K \to K$

s.t. $\Omega^\infty(-) = \psi^k$?

$$\widetilde{KU}^0(BU \times \mathbb{Z}) \longrightarrow \widetilde{KU}^0(\Omega^2 BU \times \mathbb{Z}) \cong \widetilde{KU}^0(BU \times \mathbb{Z})$$

$$\tilde{K}^0(X) \cong \tilde{K}^2(\Sigma^2 X) \xleftarrow{\cong} \tilde{K}^2(S^2) \otimes \tilde{K}^0(X) \cong \tilde{K}^0(S^2) \otimes \tilde{K}^0(X)$$

$\psi^k \longmapsto k\psi^k$

" $\left(\psi^k, \frac{1}{k}\psi^k, \cdots\right)$ " $\in \varprojlim K^0(BU \times \mathbb{Z})$

$\Rightarrow$ only get $\psi^k \in K^0K[1/k]$.

Aside: What are coh operations good for?

Coh operations detect maps between spheres:

e.g. H.I.

e.g. Steenrod ops

- HI 1 are all stable

- Indecomposables $\longleftrightarrow$ restrict dimensions to $2^i - 1$

Adams-Atiyah - pf of Hopf invariant 1

$$0 \to \tilde{K}^0(S^{4n}) \longrightarrow \tilde{K}^0(C\alpha) \longrightarrow \tilde{K}^0(S^{2n}) \to 0$$

$$L_{4n} \longmapsto y$$

$$x \longmapsto L_{2n}$$

$x \longmapsto c_{2n}$

$x^2 = y + \alpha x$

$\psi^2(x) = 2^n x + \beta_2 y$

$\psi^2(x) \equiv x^2 \mod 2$

$\Rightarrow \psi^2(x) \equiv y + \alpha x \mod 2$

$\Rightarrow \beta_2 \equiv 1 \ (2)$

$\psi^k(x) = k^n x + \beta_k y$

$$\psi^2 \psi^k x = \psi^2(k^n x + \beta_k y) = (2k)^n x + k^n \beta_2 y + 2^n \beta_k y$$

$$\|$$

$$\psi^k \psi^2 x = (2k)^n x + 2^n \beta_k y + k^{2n} \beta_2 y$$

i.e. $2^n \beta_k + k^{2n} \beta_2 = k^n \beta_2 + 2^{2n} \beta_k$

$$k^n(k^n - 1)\beta_2 = 2^n(2^n - 1)\beta_k$$

$\nu_2(n)$	n	$3^n - 1$	$\nu_2(3^n-1)$
0	1	2	1
1	2	8	3
0	3	26	1
2	4	80	4

$$\nu_2(3^n - 1) = \begin{cases} 1, & n \text{ odd} \\ \nu_2(n) + 2, & n \text{ even} \end{cases}$$

$$\nu_2(LHS) = \begin{cases} 1, & n \text{ odd} \\ \nu_2(n) + 2, & n \text{ even} \end{cases}$$

$$\nu_2(RHS) \geq n$$

So $n \leq \begin{cases} 1 & n \text{ odd} \\ \nu_2(n) + 2 & n \text{ even} \end{cases}$

$n = 2^i$

$\Rightarrow$

i	n	$\nu_2(n)+2$
1	2	3
2	4	4
3	8	5
4	16	
5	32	

No HI 1 class in $\pi^S_{2^i - 1}$, $i \geq 4$

E = homotopy commutative ring spectrum

Cooperations $\quad E_*E$ flat $/E_*$

Lemma for any X

$$E_*(E \wedge X) \xleftarrow{\cong} E_*E \otimes_{E_*} E_*X$$

(Pf) You might think

"$\mathrm{Tor}_{E_*}(E_*E, E_*X) \Rightarrow \pi_* E \wedge E \wedge X$" collapses

but this is generally true

- true for X sphere
- Inductively true for finite CW spectra
- true for all CW spectra. $\square$

Consequence: $\quad E \wedge E = E \wedge S \wedge E \longrightarrow E \wedge E \wedge E$

Consequence:

$$E \wedge E = E \wedge S \wedge E \longrightarrow E \wedge E \wedge E$$

$$E_*E \longrightarrow E_*E \otimes_{E_*} E_*E$$

E_*E is a coalgebra over E_*

(counit $E_*E \to E_*$)

E_*E is an E_*-algebra in two ways.

$$E_*E \xrightarrow[c]{} E_*E \quad \text{conjugation}$$

(E_*, E_*E) is a Hopf algebroid.

(Ref in terms of representable functors)

E_*X is an E_*E-comodule

my $+$/$-$ represents

$$H\mathbb{F}_{2*}H\mathbb{F}_2 = \mathbb{F}_2[\xi_1, \xi_2, \ldots] \qquad |\xi_i| = 2^i - 1 \qquad \xi_k = (Sq^{2^{k-1}} Sq^{2^{k-2}} \cdots Sq^2 Sq^1)^*$$

e.g. $H\mathbb{F}_{p*}H\mathbb{F}_p = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \ldots]$

$$\xi_k = (P^{p^{k-1}} P^{p^{k-2}} \cdots P^1)^*$$

$$|\tau_i| = 2p^i - 1$$

$$|\xi_i| = 2(p^i - 1) \qquad \tau_k = (P^{p^{k-1}} P^{p^{k-2}} \cdots P^p \beta)^*$$

$$\psi(\xi_i) = \sum_{i_1+i_2} \xi_{i_1}^{p^{i_2}} \otimes \xi_{i_2}$$

$$\psi(\tau_i) = \sum_{i_1+i_2=i} \xi_{i_1}^{p^{i_2}} \otimes \tau_{i_2}$$

KU_0KU: torsion-free $\qquad KU_*KU \hookrightarrow KU_*KU_{\mathbb{Q}}$

(e.g. KU_*BU torsion-free) $\qquad \cong \mathbb{Q}[u^{\pm 1}, v^{\pm 1}]$

Lemma: $\pi_* E \wedge F_{\mathbb{Q}} \cong \pi_* E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \pi_* F_{\mathbb{Q}}$

$\cong HQ_*(E \wedge F)$

$KU_* KU \cong K_* \otimes_{K_0} K_0 KU$

Thm (Adams - Harris)

$$K_0 K \hookrightarrow K_0 K_{\mathbb{Q}} \cong \mathbb{Q}[w^{\pm}] \qquad w = \frac{v}{u}$$

$$K_0 K_{\mathbb{Q}} \xrightarrow{1 \wedge \psi^k} K_0 K_{\mathbb{Q}} \xrightarrow{\mu} (K_0)_{\mathbb{Q}} \cong \mathbb{Q}$$

$$w^i \longmapsto k^i w \longmapsto k^i$$

$$K_0 K_{\mathbb{Q}} \otimes K^0(K)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$$

$$(f, \psi^k) \longmapsto f(k)$$

$$\Rightarrow \quad f(w) \longmapsto f(k)$$

$$K_0 K \cong \left\{ f(w) \in \mathbb{Q}[w^{\pm}] \;\middle|\; f(k) \in \mathbb{Z}[1/k] \quad \forall k \right\}$$

"functions on Adams operations"

More clear if $K_0 K_{(p)}$ $\qquad K_0 K_{(p)} = \{ f(w) \in \mathbb{Q}[w^{\pm}] \mid f(k) \in \mathbb{Z}_{(p)},\ p \nmid k \}$

[Question: what is $K^0 K$?]