2 - the stable homotopy category

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Recall Want SHC

$$\begin{tikzcd} SW \arrow[r] \arrow[rr, bend left] & SHC \arrow[r] & \text{Coh. th'y's} \\ SW_f \arrow[u] \arrow[ur, "\text{full}"'] & & \end{tikzcd}$$

$SHC = Sp[\text{stable equiv}^{-1}]$

Sp: $X = \{\underline{X}_i, \sigma_i\}$ $\quad \Sigma X_i \to X_{i+1}$ $\quad X = \text{"}\varinjlim \Sigma^{-i} \underline{X}_i\text{"}$ Complete Cocomplete

maps $f_i : X_i \to Y_i$ ... e.g. Wedges, products

(X, n) $\quad (X,n): * \quad \cdots \quad * \quad \underset{n}{X} \quad \underset{n+1}{\Sigma X} \quad \cdots$

Coh th'y $\tilde{E}^k(X) = [X, E_k]_*$, $E_k \xrightarrow{\simeq} \Omega E_{k+1}$ "Ω-spectrum"

(e.g. $H\pi$ $\quad \underline{H\pi}_n = K(\pi, n)$)

$\pi_k^s X := \varinjlim \pi_{k+n}(\underline{X}_n) = \pi_{k+n}(\underline{X}_n)$ if X Ω-spectrum

f iso in SW_f $\Longleftrightarrow$ $f_* : \pi_*^s X \to \pi_*^s(Y)$ iso

$f : X \to Y$ in Sp $\Longleftrightarrow$ $f_* : \tilde{X}^*(-) \to \tilde{Y}^*(-)$ iso
π_*^s-iso

Motivation: $SHC := Sp[\text{st. equiv}^{-1}]$ — how to perform?

Recall: $Top[w.e.^{-1}]$
- CW - approximation
- Whitehead thm

$Top[w.e.^{-1}] := CW/\simeq$

Strategy Spectra:
- homotopy equivalences $\Downarrow$
- levelwise w.e.'s (l.w.e.) $\Downarrow$
- Stable equiv.

(1) every spectrum is levelwise equiv. to CW spectrum

(2) $\left(f \text{ l.w.e. between CW spectra}\right) \iff \left(f \text{ h.e.}\right)$

$$Sp[\text{l.w.e.}^{-1}] := CWSp/\simeq$$

(3) every spectrum is st. equiv. to Ω-spectrum.

$$Sp[\text{st.equiv.}^{-1}] := CW\Omega Sp/\simeq$$

<u>Homotopy</u> $K \in Top_*$, $X \in Sp$ $\Longrightarrow$ $X \wedge K$ $\qquad \underline{X \wedge K}_i = \underline{X}_i \wedge K$

$$\Sigma \underline{X}_i \wedge K \xrightarrow[\sigma_i \wedge 1]{} \underline{X}_{i+1} \wedge K$$

<u>dually</u> X^K $\qquad \underline{X^K}_i = \underline{Map}_*(K, X)$

Note: $X \wedge S^0 \cong X \cong X^{S^0}$

<u>Check</u>: $- \wedge K : Sp \rightleftarrows Sp : (-)^K$

<u>Def</u> $f, g : X \to Y$ <u>homotopic</u> if

$$\begin{tikzcd} X \arrow[d, "i_0"] \arrow[dr] & \\ X \wedge I_+ \arrow[r, "H"] & Y \\ X \arrow[u, "i_1"'] \arrow[ur] & \end{tikzcd}$$

(<u>Check</u> collection of homotopies $f_i \simeq g_i$)

<u>Def</u> $f : X \to Y$ h.e. if $\exists\ g : Y \to X$ s.t. $f \circ g \simeq Id$, $g \circ f \simeq Id$

<u>Note</u>: can also define $f : X \to Y$

- $C(f)$ $\qquad \underline{C(f)}_i = C(f_i)$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ X \wedge I \arrow[r] & C(f) \end{tikzcd}$$

- $F(f)$

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \arrow[r, "f"] \arrow[ur, phantom, "\lrcorner"] & Y \end{tikzcd}$$

Mapping cylinders, mapping path space,

K CW spectrum $\Rightarrow$ can build K inductively from (S^k, i)

CW properties: $K = \varinjlim K^{[n]}$ $\qquad K^{[n]} = \{ \underline{K}_i^{[n+i]} \}$

$$K^{[n-1]} \longrightarrow K^{[n]} \longrightarrow \underbrace{\bigvee (S^{n+i}, i)}_{\text{stable } n\text{-cells}}$$

$$K^{[n-1]} \longrightarrow K^{[n]} \longrightarrow \underset{\substack{\text{stable} \\ n\text{-cells}}}{\bigvee} (S^{n+i}, i)$$

cellular perspective: build $\underline{K}_0$, then build $\underline{K}_1$, etc....

$$\begin{tikzcd}
(\bigvee S^n, i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (\underline{K}_0, \dots, \underline{K}_{i-1}, \underline{K}_i^{[n-1]}) \arrow[d] \\
(\bigvee D^n, i) \arrow[r] & (\underline{K}_0, \dots, \underline{K}_i^{[n]})
\end{tikzcd}$$

Def $f: X \to Y$ is levelwise w.e.

if $f_i : X_i \longrightarrow Y_i$

are all w.e.'s

Note h.e. $\Rightarrow$ l.w.e. $\Rightarrow$ π_*-equiv.

Prop $X \in Sp$
(CW approx)

$\exists\ \tilde{X} \xrightarrow[f]{} X$, $\tilde{X}$ is a CW spectrum

f is a levelwise w.e.

(pf) Relative CW approx

$$\begin{tikzcd}
\overset{\text{cw}}{A} \arrow[r, hook] \arrow[d, hook, "\text{inclusion of cw cx's}"'] & \overset{\text{cw}}{B} \\
\tilde{B} \arrow[ur, "\text{w.e.}"'] &
\end{tikzcd}$$

$\tilde{X}_0 \longrightarrow X_0$

$$\begin{tikzcd}
& \Sigma_1 X_i \arrow[dr] & \\
\Sigma_1 \tilde{X}_i \arrow[ur] \arrow[dr] & & X_{i+1} \\
& \tilde{X}_{i+1} \arrow[ur] &
\end{tikzcd}$$

Thm (levelwise Whitehead thm for spectra)

$f: X \to Y$ levelwise w.e.

X, Y CW sp.

$\Longrightarrow f$ is a h.e.

- Cofibrant spectrum $\Sigma_1 X_i \longrightarrow X_{i+1}$ is a pointed cofibration

$$X^c \xrightarrow[\text{h.e.}]{} X$$

- lemma: "Weak map"

$$\left.\begin{array}{l} f_i: \underline{X}_i \longrightarrow \underline{Y}_i \\ X \text{ cofibrant} \\ \begin{tikzcd} \Sigma_1 \underline{X}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & \Sigma_1 \underline{Y}_i \arrow[d] \\ \underline{X}_{i+1} \arrow[r] & \underline{Y}_{i+1} \end{tikzcd} \end{array}\right\} \Longrightarrow \begin{array}{l} \exists\ \tilde{f}_i \simeq f_i \\ \text{s.t. maps honestly commute} \end{array}$$

$$\begin{tikzcd}
& B \arrow[r] & D \\
\arrow[ur] & & \arrow[u]
\end{tikzcd}$$

$A \longrightarrow B$

$X_{i+1} \longrightarrow Z_{i+1}$

Note weak map induces map on π_*^s

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \Longrightarrow \begin{tikzcd} & B \arrow[r] & D \\ A \arrow[ur] \arrow[rr, "h"] \arrow[dr] & & \mathrm{Map}_*(I_+ D) \arrow[u] \arrow[d] \\ & C \arrow[ur, dashed] \arrow[r] & D \end{tikzcd}$$

Ω-approximation X spectrum $\Sigma X_i \to X_{i+1}$

$\Longrightarrow$ $X \xrightarrow[\text{st. equiv.}]{} \omega X$ $\quad$ ωX is an Ω-spectrum

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(pf) Define $\underline{\omega X_i} = \varinjlim_k \Omega^k X^c_{k+i}$ $\square$

$X \in Sp$

$$X \xleftarrow[\text{l.w.e.}]{} \tilde{X} \xrightarrow[\text{St. equiv}]{} \omega\tilde{X} \xleftarrow[\substack{\text{l.w.e.} \\ \text{(h.e.)}}]{} \widetilde{(\omega\tilde{X})}$$

Ω-CW spectrum

Thm: up to zig-zag of stable equivalences

every spectrum $\simeq$ ΩCW

Prop: $f: X \to Y$ $\quad$ X, Y Ω-spectra

$\Longrightarrow f$ levelwise equiv.

~~Stable~~ Whitehead Thm $f: X \to Y$ $\quad$ X, Y ΩCW

$\Longrightarrow f$ h.e.

(follows from previous prop and levelwise whitehead)

Γ $\quad$ 7^s $\quad$ ... $\quad$ SHC. $\quad$ ...

$[-,-]^s$ denotes maps in SHC.

$SHC \longrightarrow$ Coh. Thy's

full & essentially surjective

Not faithful.

- SNT $\Rightarrow$ map of spectra

- phantom maps. "f-phantom maps"

[Exercise: Read and understand an example of phantom map]