

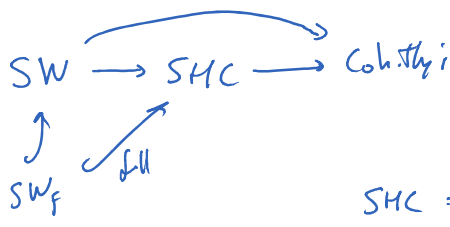
2 - the stable homotopy category

Tuesday, September 2, 2014 12:04 PM

Recall

Want

SHC



$$SHC = Sp[\text{stable equiv}]$$

$$\text{Sp}: X = \{X_i, \sigma_i\}$$

$$\sum X_i \rightarrow X_{i+1}$$

$$X = \varinjlim \sum^{-i} X_i$$

Complete
Locomplete

maps $f_i: X_i \rightarrow X_{i+1}$...

e.g. Weibier, products

$$(X, n)$$

$$(X, n) = * \rightarrow \dots \rightarrow * \rightarrow X_n$$

$$\Sigma X_{n+1}$$

$$\left(\begin{array}{l} \text{e.g. } H\pi \\ H\pi_n = k(\pi, n) \end{array} \right)$$

(oh thly) $\tilde{E}^k(X) = [X, E_k]_*$, $E_k \xrightarrow{\cong} \Omega E_{k+1}$

$$\pi_k^s X := \varinjlim \pi_{k+n}(X_n) \quad \text{"}\Omega\text{-spectrum"}$$

↖ if X Ω -spectrum

$$f \text{ iso in } SW_f \iff f_*: \pi_*^s X \rightarrow \pi_*^s Y \quad \text{iso}$$

$$f: X \rightarrow Y \text{ in } Sp \iff f_*: \tilde{X}^*(-) \rightarrow \tilde{Y}^*(-) \quad \text{iso}$$

$\pi_*^s \rightarrow \pi_*$

Modification: SHC := Sp[st. equiv] can have to perform?

Recall: Top[w.e.]

- CW - approximations
- Whitehead thm

$$\text{Top[w.e.]} := CW / \cong$$

Strategy

- Spectrum:
- homotopy equivalence
 - local w.e.'s (l.w.e.)
 - stable equiv.

(1) every spectrum is local equiv. to CW spectrum

a) (f h.e. between CW spectra) \Leftrightarrow (f h.e.)

$$Sp[l.w.e.] := CWSp / \cong$$

(3) every spectrum is st. equiv. to Ω -spectrum.

$$Sp[st.equiv.] := CW\Omega Sp / \cong$$

Homotopy

$$K \in Top, \\ X \in Sp$$

$$\Rightarrow X \wedge K$$

$$\underline{X} \wedge K_i = \underline{X}_i \wedge K$$

$$\sum_i \underline{X}_i \wedge K \xrightarrow{\sigma_i^{-1}} \underline{X}_{i+1} \wedge K$$

dualy X^k

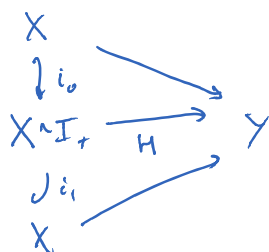
$$\underline{X}_i^k = Map_s(K, X)$$

Note: $X \wedge S^0 = X \cong X^{S^0}$

Check: $\wedge K: Sp \rightleftharpoons Sp(-)^k$

Def

$f, g: X \rightarrow Y$ homotopic if



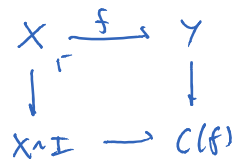
(Check collection of homotopies $f_i \cong g_i$)

Def $f: X \rightarrow Y$ h.e. if $\exists g: Y \rightarrow X$ s.t. $f \circ g = Id, g \circ f \simeq Id$

Note: can also define $f: X \rightarrow Y$

* $C(f)$ $C(f)_i = C(f_i)$

* $F(f)$ $F(f) \rightarrow Y^I$



Mapping Cylinders, mapping path space, $X \xrightarrow{f} Y$

K CW spectrum \Rightarrow can build K inductively from (S^k, i)

CW perspective:

$$K = \varinjlim K^{[n]}$$

$$K^{[n]} = \{ \underline{K}_i^{[n+i]} \}$$

$$K^{[n-1]} \rightarrow K^{[n]} \rightarrow V(S^{n+i}, i)$$

finite n-cells

$$K^{(n-1)} \rightarrow K^{(n)} \rightarrow V(S^{n,i})$$

finite
n-cells

cellular perspective: build \underline{K}_0 , then build \underline{K}_1 , etc....

$$\begin{array}{ccc} V(S^n, i) & \longrightarrow & (K_0, \dots, K_i, K_i^{(n-1)}) \\ \downarrow \lrcorner & & \downarrow \\ (V D^n, i) & \longrightarrow & (K_0, \dots, K_i^{(n-1)}) \end{array}$$

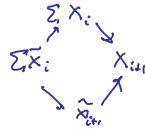
Def $f: X \rightarrow Y$ is levelwise w.e.
 if $f_i: X_i \rightarrow Y_i$
 on all w.e.'s

Note $w.e. \Rightarrow l.w.e. \Rightarrow H. equiv.$

Prop (CW approx)

$\exists \tilde{X} \xrightarrow{f} X$, \tilde{X} is a CW spectrum
 f is a levelwise w.e.

(sketch) Relative CW approx $\tilde{X}_0 \rightarrow X_0$
 $A \rightarrow B$
 $\downarrow \quad \uparrow$
 $B \rightarrow A$
rel. CW approx



Thm (Whitehead thm for spectra)

$f: X \rightarrow Y$ levelwise w.e.
 X, Y CW sp.
 $\Rightarrow f$ is a h.e.

• Coherent spectrum $\sum_i X_i \rightarrow X_{i+1}$ is a pointed cofibration

$$\begin{array}{ccc} X^c & \longrightarrow & X \\ \uparrow h.e. & & \end{array}$$

• Lemma $f_i: X_i \rightarrow Y_i$

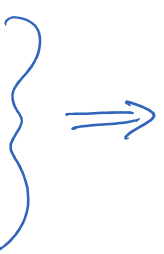
X coherent

$$\exists \tilde{f}_i = f_i$$

"Weak map"

$$\begin{array}{ccc} \sum_i X_i & \longrightarrow & \sum_i Y_i \\ \downarrow \hookrightarrow & & \downarrow \hookrightarrow \\ X_{i+1} & \longrightarrow & Y_{i+1} \end{array}$$

s.t. maps levelwise coherent



$$A \rightarrow B$$

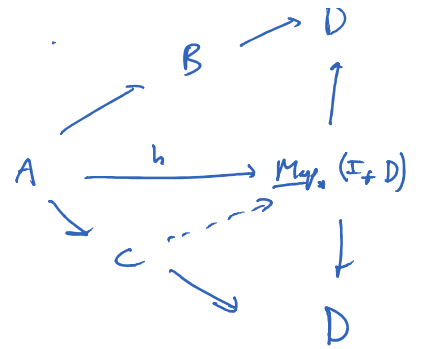
$$\begin{array}{ccc} & & D \\ & \nearrow & \uparrow \\ & B & \end{array}$$

$$\Delta_{it_1} \rightarrow \Delta_{it_2}$$

Note weak map

induces map on π_n^s

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array} \Rightarrow$$



Ω -approximate

$$X \text{ spectra } \Sigma X_i \rightarrow X_{it_1}$$

$$\Rightarrow X \xrightarrow{\gamma} \omega X$$

st. equiv.

ωX is an Ω -spectrum

(pf) Define $\omega X_i = \lim_k \Omega^k X_{-k+i}$

$X \in Sp$

$$X \xleftarrow{\text{l.w.e.}} \tilde{X} \xrightarrow{\text{st. equiv.}} \omega \tilde{X} \xleftarrow{\text{l.w.e. (h.e.)}} \widetilde{(\omega \tilde{X})}$$

\uparrow Ω -CW spectrum

Thm: up to Ω -equiv of stable equivalences

every spectrum is Ω CW

Prop: $f: X \rightarrow Y$ X, Y Ω -spectrum $\Rightarrow f$ homotopy equiv.

Stable Whitehead Thm

$$f: X \rightarrow Y \quad X, Y \text{ } \Omega\text{CW}$$

$$\Rightarrow f \text{ h.e.}$$

(follows from previous prop and homotopy whitehead)

Γ γ^s \dots is SHC. \dots

$[-, -]^s$ denotes maps in SHC.

SHC \rightarrow Coh. th'y's

full & essentially surjective
Not faithful.

• SNT \Rightarrow map of spectra

• phantom maps. "f-phantom maps"

[Exercise: Read and understand an example of phantom map]