PSET 3

ASSIGNED 9/16/16, "DUE": 9/23/16

As always, the "basic" problems are required.

1. (Basic) Suppose that $V \to X$ and $W \to Y$ are orientable vector bundles over X and Y, respectively. Show that if $[V] \in H^n(V, V - 0)$ is a Thom class for V, and $[W] \in H^m(W, W - 0)$ is a Thom class for W, then the relative cup product

$$[V] \cup [W] \in H^{n+m}(V \times W, V \times (W - 0) \cup (V - 0) \times W)$$

restricts to a Thom class for $V \times W \to X \times Y$.

2. (Basic) Deduce that $e(V \oplus W) = e(V) \cup e(W)$. Then if V is a vector bundle with a non-vanishing section, then the Euler class $e(V)$ must vanish.

3. (Basic) Let

$$t : (\mathbb{C}P^\infty)^{\times n} \to BU(n)$$

classify the sum $L_1 \oplus \cdots \oplus L_n$ of the n different cannonical line bundles. Deduce from the Whitney sum formula that under the map

$$t^* : \mathbb{Z}[c_1, \dots, c_n] = H^*(BU(n)) \to H^*((\mathbb{C}P^\infty)^n) = \mathbb{Z}[x_1, \dots, x_n]$$

we have

$$t^* c_i = \sigma_i(x_1, \dots, x_n)$$

where σ_i is the ith elementary symmetric polynomial (Hint: $x_i = c_1(L_i)$). Deduce that the map d^* is an injection, and gives an isomorphism

$$H^*(BU(n)) \cong \{\text{symmetric polynomials}\} \subseteq \mathbb{Z}[x_1, \dots, x_n].$$

4. (Basic) Look at the discussion at the end of appendix C of Milnor-Stasheff. Why does the bundle

$$V = (\widetilde{M} \times \mathbb{R}^n)/\Pi \to M$$

admit a flat connection? Milnor-Stasheff continue to give an example of such a bundle with $e(V) \neq 0$. Why does this imply, as they assert, that, unlike Chern classes, there is not a curvature representation of Euler class for all bundles which is independent of connection?

Gysin maps

(Less basic) The next two problems investigate a map which goes the "wrong way" in cohomology called the Gysin map. From now on we *always* work with homology with mod 2 coefficients to avoid having to discuss orientations, and manifolds are assumed to be smooth, connected, closed, and compact.

Let $i : N \hookrightarrow M$ be the inclusion of a submanifold of a manifold M, with $\dim N = n$ and $\dim M = m$. Give the tangent bundle TM a metric, and define $\nu = TN^\perp$

to be the normal bundle of N in TM. The "tubular neighborhood theorem" of differential topology asserts that there is a tubular neighbor $\mathrm{Tube}(N)$ of N in M whose closure $\overline{\mathrm{Tube}}(N)$ is diffeomorphic to the disk bundle $D(\nu)$. Let

$$P : M \to \overline{\mathrm{Tube}}(N)/\partial\overline{\mathrm{Tube}}(N) \approx N^\nu$$

be the map which sends all points outside of $\mathrm{Tube}(N)$ to the basepoint. This map is called the *Pontryagin-Thom* collapse map. It induces, via the Thom isomorphism, a map going in the wrong way called a *Gysin* map:

$$i_! : H^*(N) \cong \widetilde{H}^{*+m-n}(N^\nu) \xrightarrow{P^*} H^{*+m-n}(M).$$

In particular, we get a (mod 2) cohomology class $[N]$ whose dimension is the codimension of N in M:

$$[N] := i_!(1) \in H^{m-n}(M).$$

5. Verify that for the inclusion of a point $* \hookrightarrow M$, the class $[*] \in H^m(M)$ is dual to the fundamental class $[M] \in H_m(M)$.

6. A pair of submanifolds N_1 and N_2 of dimensions n_1 and n_2, respectively, are said to be *transverse* in M if for each point $x \in N_1 \cap N_2$, the tangent space TM_x is spanned by the subspaces $(TN_1)_x$ and $(TN_2)_x$. The implicit function theorem then may be used to show that $N_1 \cap N_2$ is a submanifold of dimension $n_1 + n_2 - m$, with tangent bundle $TN_1 \cap TN_2 \hookrightarrow TM$.

Verify the formula

$$[N_1] \cup [N_2] = [N_1 \cap N_2] \in H^{2m-n_1-n_2}(M).$$

In other words, for geometric cocycles in general position, the cup product is given by intersection.