Kiran: Crystals + Crystalline Site

Note Title 2/19/2009

[Following Berthelot - Ogus]

Goal: Develop a meaningful form of de Rham cohomology, not restricted to characteristic zero.

60's Grothendieck wants to prove Weil conj's

"Lefschetz trace formula on H^*"

2 schools of thought:

- ℓ-adic cohom
- de Rham coh (Dwork)

proved rationality of zeta functions "prematurely"

Basic idea: Infinitesimal site for

$X/_k$ char $k = 0$

Objects: $U \hookrightarrow T$

open in X

infinitesimal thickening

F = sheaf on this site

$R^*\Gamma F$ = "de Rham coh"

Aside: why doesn't this work in char p?

Can't integrate: Poincaré lemma fails

e.g. alg de Rham coh of $\mathbb{A}^1_{\mathbb{F}_p}$

$$0 \to \mathbb{F}_p[x] \xrightarrow{d} \mathbb{F}_p[x]\{dx\} \longrightarrow 0$$

$$f \longmapsto \frac{df}{dx}\, dx$$

Not surjective!

$\frac{x^{p-1}}{(p-1)!}$ is not in the image of d

So $H^1_{dR}(A_{\mathbb{F}_p}) \neq 0$

actually not f.g. as $\mathbb{Z}$-mod!

<u>Same problem for $\mathbb{Z}_p$!!</u>

examples of infinitesimal thickenings!

$$\mathbb{F}_p \longleftarrow \mathbb{F}_p[\varepsilon]/\varepsilon^2$$

$$\mathbb{F}_p \longleftarrow \mathbb{Z}/p^2$$

"char 0 thickening"

Rather replace:

$$\mathbb{Z}_p[x] \rightsquigarrow \mathbb{Z}_p\left[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots\right]$$

BAD GOOD

$$\int \sum_{i=0}^{\infty} c_i \frac{x^i}{i!} = \sum_{i=1}^{\infty} c_{i-1} \frac{x^i}{i!}$$

Need "divided powers"

$A =$ any

$I \subset A$ ideal

A divided power structure ("PD structure") on (A, I) is a collection of maps

$$\gamma = (\gamma_i) \quad i \geq 0$$

$$\gamma_0(x) = 1, \quad \gamma_i : I \to I$$

$$\left(\text{idea: } \gamma_i(x) = \frac{x^i}{i!} \right)$$

Axioms:

$$\gamma_0(x) = 1$$

$$\gamma_1(x) = x$$

$$\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$$

$\lambda \in A$
$$\gamma_k(\lambda x) = \lambda^k \gamma_k(x)$$

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x)$$

"Most fun one" $\rightsquigarrow$
$$\gamma_i(\gamma_j(x)) = \underbrace{\frac{(ij)!}{(i)!(j!)^i}}_{\text{integer}} \gamma_{ij}(x)$$

Morphisms of PD rings:

$$(A, I, \gamma) \longrightarrow (B, J, \delta)$$

- $f: A \to B$ ring hom
- $f(I) \subseteq J$
- $\delta_n(f(x)) = f(\gamma_n(x))$

Examples of P.D. rings

$(\mathbb{Z}_p, (p))$

$(A, (0))$ A any ring

If $\mathbb{Q} \subseteq A$, then (A, I) has unique PD structure

$$\gamma_i(x) = \frac{x^i}{i!}$$

$(\mathbb{Z}/p^n, (p))$ possesses canonical P.D. structure coming from $(\mathbb{Z}_p, (p))$

BUT there exist other bizarre P.D. structures

More Generally

If V = DVR mixed char $(0, p)$

e = absolute ramification index

$$(p) = (\pi)^e$$

Get P.D. structure on $(V, (\pi))$

as long as $e \leq p-1$

Divided power envelope

(B, J) maybe does not have divided powers

(A, I, γ) P.D. structure

$\left.\begin{array}{l} B = A\text{-algebra} \\ J = \text{Ideal of } B \end{array}\right\}$ want P.D. struct

e.g. $(\mathbb{Z}_p, (p)) \longmapsto (\mathbb{Z}_p[x], (p, x))$

Then $\exists$ B-alg $\left(D_{B,\gamma}(J), \overset{\text{containing } J}{\bar{J}}, \delta\right)$

s.t. • δ compatible w/ γ

• universal:

$$\begin{tikzcd}
(A, I, \gamma) \arrow[rr] \arrow[d] & & (C, K, \varepsilon) \\
(B, J) \arrow[urr] \arrow[d] & & \\
(D_{B,\gamma}(J), \bar{J}, \delta) \arrow[uurr, dashed, "\exists!"'] & &
\end{tikzcd}$$

<u>e.g.</u>

NOT NOETH!

$$\left(\mathbb{Z}_p[x], (p, x)\right) \longmapsto \left(\sum_{i=0}^{\infty} c_i \frac{x^i}{i!}, \quad \begin{array}{l} c_i \in \mathbb{Z}_p \\ \text{a.a.} \\ c_i = 0 \end{array}\right)$$

$$\cap$$

$$\mathbb{Q}_p[x]$$

$$\left(\mathbb{Z}_p[[x]],\ (p,x)\right) \longmapsto \left(\sum_{i=0}^{\infty} c_i \frac{x^i}{i!},\ c_i \in \mathbb{Z}_p\right)$$

$$\cap$$

$$\mathbb{Q}_p[[x]]$$

---

e.g. formal completion $\hat{X}_Y$

$$Y \xhookrightarrow{\text{closed}} X \text{ scheme}$$

$$\downarrow$$

"P.D. completion"

Could make Poincaré lemma hold.

But! If we are thirsty for blood:

CRYSTALLINE SITE

(S, I, γ) = base P.D. scheme

(commonly equal to $(\mathbb{Z}/p^n, (p), \text{canonical})$)

Assume $\exists$ extension of P.D. structure to X

$$\begin{tikzcd} X \arrow[d] \\ S \end{tikzcd} \quad \text{scheme}$$

[Typically

$$X \xrightarrow[\text{smooth}]{} \mathrm{Spec}(\mathbb{F}_p) \longrightarrow \mathrm{Spec}(\mathbb{Z}/p^n)$$
]

$\mathrm{Cris}(X/S)$ = site consisting of

<u>Objects</u>: closed S-immersion, nil test $T \to S$ comp. w/ P.D. structure

$$(U \hookrightarrow T, \delta)$$

Zariski open in X

P.D. structure on ideal of closed immersion

<u>Morphisms</u>: squares which commute, induce

Coverings:

$$\begin{array}{c} (U_i \hookrightarrow T_i, \delta_i) \\ \downarrow \\ (U \hookrightarrow T, \delta) \end{array}$$

If U_i cover U Zariski-wise

Can talk about sheaves on this site.

$$\mathcal{F} \in Sh(Cris(X/S), Ab)$$

Assume S = affine

$$R^* \Gamma \mathcal{F}$$

What is Γ? don't have final object!

General answer

Site $\rightsquigarrow$ Topos

$\mathcal{C}$ $\qquad$ $Sh(\mathcal{C}, Sets)$

$$\Gamma \mathcal{F} = Morph(e, \mathcal{F})$$

e = final object of Topos

= constant sheaf with singleton output

Rmk: Γ is difficult to compute in $Sh(Cris(X/S))$

But if you have a good test pro-object:

e.g.

$$Spec(\mathbb{F}_p[x]) \longrightarrow Spec(\mathbb{Z}/p^n)$$

Good test object: $\mathbb{Z}/p^n[x]$

De Rham-Witt complex:

Furnishes a suitable test object.

Need Functoriality:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

Would like morphism of sites:

giving $\quad g^*: Sh(Cris(X/S)) \longrightarrow Sh(Cris(X'/S'))$

Would like $\quad g_{cris}: Cris(X'/S') \longrightarrow Cris(X/S)$

DOES NOT EXIST

Why do we thing we can fix this?

Groth knew something:

Comes out of work of Dwork

(Monsky - Washnitzer)

$$\begin{array}{c} X \\ \downarrow f \\ \mathrm{Spec}(k) \end{array}$$

k perfect, char $= p$

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[dr, "f_1"'] & & \mathfrak{X}_2 \arrow[dl, "f_2"] \\ & W(k) & \end{tikzcd}$$

two lifts

$\exists$ canonical iso:

$$H^*_{dR}(X_1) \cong H^*_{dR}(X_2)$$

$$X_i = (\mathfrak{X}_i)_F \qquad F = W(k) \otimes \mathbb{Q}$$

(Actually $H^*_{dR}(\mathcal{X}_1) \cong H^*_{dR}(\mathcal{X}_2)$)

Groth: constructed morphism directly on the topos.

(g^*, g_*)

[topos thy g^* nice $\rightsquigarrow$ get g_* as Right Kan extension]

Caveat: $(U, T, \delta) \in \mathrm{Cris}(X/S)$

Consider sheaf $\tilde{T} = \mathrm{Hom}(-, (U, T, \delta))$

$(g^* \tilde{T}) : (U', T', \delta') \longmapsto$

$\mathrm{P.D.Hom}_g((U', T', \delta'), (U, T, \delta))$

$$\mathrm{P.D.Hom}_g\big((U',T',\delta'),(U,T,\delta)\big) := \begin{cases} \emptyset, & g(U') \not\subseteq U \\ \{h : (T',\delta') \to (T,\delta)\} \end{cases}$$

$$\begin{tikzcd}
U' \arrow[rr] \arrow[dr, hook] \arrow[dd] & & U \arrow[dr, hook] \arrow[dd, "h"] & \\
& T' \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
S \arrow[rr, "g"'] & & S &
\end{tikzcd}$$

$\mathrm{Cris}(X/S)$ has a special object

<u>Structure</u> <u>Sheaf</u>: $\mathcal{O}_{(X/S)_{cris}}$

$$(U,T,\delta) \longrightarrow \Gamma\mathcal{O}_T$$

Crystalline cohomology: $H^*_{cris}(X/S) := R^*\Gamma\,\mathcal{O}_{(X/S)_{cris}}$