

# Kiran: Crystals + Crystalline Site

Note Title

2/19/2009

[Following Bertelot - Ogus]

Goal: Develop a meaningful form of de Rham cohomology, not restricted to characteristic zero.

60's Grothendieck wants to prove Weil conj's  
"Lefschetz trace formula on  $M^*$ "

2 schools of thought!

•  $\ell$ -adic cohom

• de Rham coh (Dwork)

↑  
proved rationality of  
zeta function  
"prematurely"

Basic idea! Infinitesimal site for  
 $X/k$   $\text{char } k = 0$

Objects!  $U \hookrightarrow T$   
open in  $X$   $\uparrow$  infinitesimal thickening

$F =$  sheaf on the site

$R^* \Gamma F =$  "de Rham cohomology"

Aside! why doesn't this work in  $\text{char } p$ ?

Can't integrate! Poincaré lemma fails

e.g. de Rham cohomology of  $A^1_{\mathbb{F}_p}$

$$0 \rightarrow \mathbb{F}_p[x] \xrightarrow{d} \mathbb{F}_p[x]\{dx\} \rightarrow 0$$

$$f \longmapsto \frac{df}{dx} dx$$

Not Surjective!

$\frac{x^{p-1}}{(p-1)!}$  is not in the image of  $d$

So  $H_{dR}^1(A_{\mathbb{F}_p}) \neq 0$   
actually not f.g.  
as  $\mathbb{Z}$ -mod!

Same problem for  $\mathbb{Z}_p$ !!

examples of infinitesimal thickenings!

$$\mathbb{F}_p \longleftarrow \mathbb{F}_p[\varepsilon]/\varepsilon^2$$

$$\mathbb{F}_p \longleftarrow \mathbb{Z}/p^2$$

"char 0  
thickening"

Rather replace!

$$\mathbb{Z}_p[x] \rightsquigarrow \mathbb{Z}_p[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots]$$

BAD

GOOD

$$\int \sum_{i=0}^{\infty} c_i \frac{x^i}{i!} = \sum_{i=1}^{\infty} c_{i-1} \frac{x^i}{i!}$$

Need "divided powers"

$$A = \mathbb{R}[y]$$

$$\mathbb{I} \subset A \text{ ideal}$$

A divided power structure <sup>"PD structure"</sup> on  $(A, \mathbb{I})$

is a collection of maps

$$\gamma = (\gamma_i) \quad i \geq 0$$

$$\gamma_0(x) = 1, \quad \gamma_i: \mathbb{I} \rightarrow \mathbb{I}$$

$$\left( \text{ideal } \gamma_i(x) = \frac{x^i}{i!} \right)$$

Axioms!  $\gamma_0(x) = 1$

$$\gamma_1(x) = x$$

$$\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x) \gamma_j(y)$$

$$\lambda \in A \quad \gamma_k(\lambda x) = \lambda^k \gamma_k(x)$$

$$\gamma_i(x) \gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x)$$

'Most fun one'  $\rightarrow \gamma_i(\gamma_j(x)) = \frac{(ij)!}{(i)!(j!)^i} \gamma_{ij}(x)$

integer

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Morphisms of PD rings:

$$(A, \mathcal{I}, \gamma) \longrightarrow (B, \mathcal{J}, \delta)$$

•  $f: A \rightarrow B$  *my home*

•  $f(\mathcal{I}) \subseteq \mathcal{J}$

•  $\delta_n(f(x)) = f(\gamma_n(x))$

Examples of P.D. rings

$$(\mathbb{Z}_p, (p))$$

$$(A, (0)) \quad A \text{ any ring}$$

If  $Q \subseteq A$ , then  $(A, I)$  has

Unique PD structure

$$\gamma_i(x) = \frac{x^i}{i!}$$

$$(\mathbb{Z}/p^n, (p)) \quad \text{possesses canonical}$$

P.D. structure

coming from

$$(\mathbb{Z}_p, (p))$$

BUT there exist other  
bizarre P.D. structures

## More Generally

if  $V = \text{DVR}$  mixed char  $(0, p)$

$e =$  absolute ramification index

$$(P) = (\pi)^e$$

Get P.D. structure on  $(V, (\pi))$

as long as  $e \leq p-1$

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## Divided power envelope

$(B, J)$

maybe does not have  
divided powers

$(A, I, \gamma)$

P.D. structure

$B = A$ -algebra

$J =$  Ideal of  $B$

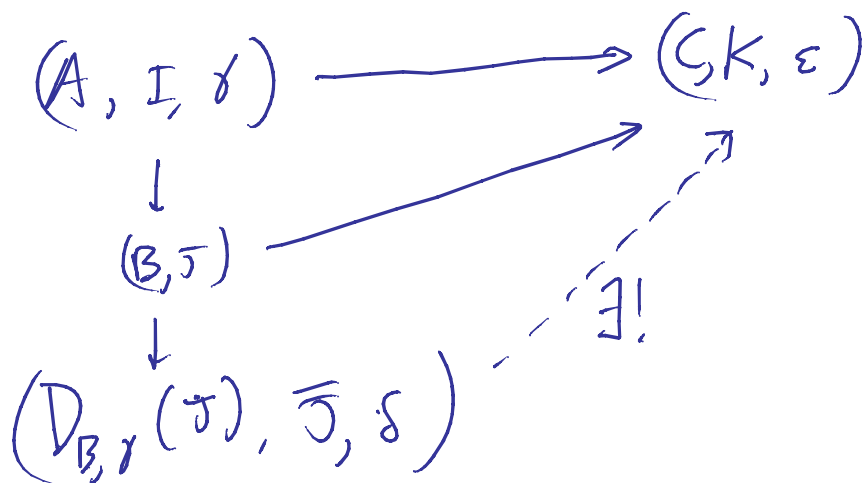
} want P.D. str.

e.g.  $(\mathbb{Z}_p, (p)) \rightsquigarrow (\mathbb{Z}_p[x], (p, x))$

Then  $\exists$  B-obj  $(D_{B,\delta}(J), \overline{J}, \delta)$  <sup>contains J</sup>

s.t.  $\delta$  compatible w/  $\delta$

& unless:



NOT NO ETH!

eg.

$$\left( \mathbb{Z}_p[x], (p, x) \right) \longmapsto \left( \sum_{i=0}^{\infty} c_i \frac{x^i}{i!}, \begin{array}{l} c_i \in \mathbb{Z}_p \\ a_{i+1} \\ c_0 = 0 \end{array} \right)$$

$$\cap \mathbb{Q}_p[x]$$

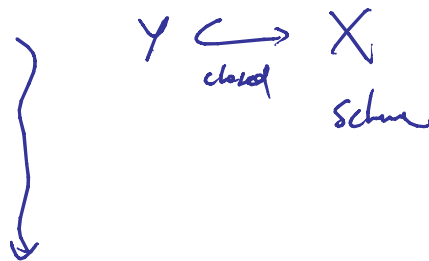


$$\left( \mathbb{Z}_p[x], (p, x) \right) \mapsto \left( \sum_{i=0}^{\infty} c_i \frac{x^i}{i!}, c_i \in \mathbb{Z}_p \right)$$

$\cap$   
 $\mathbb{Q}_p[x]$

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e.g. formal completion  $X_{\mathfrak{y}}^{\wedge}$



"P.D. completion"

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Could make Poisson lemma hold.

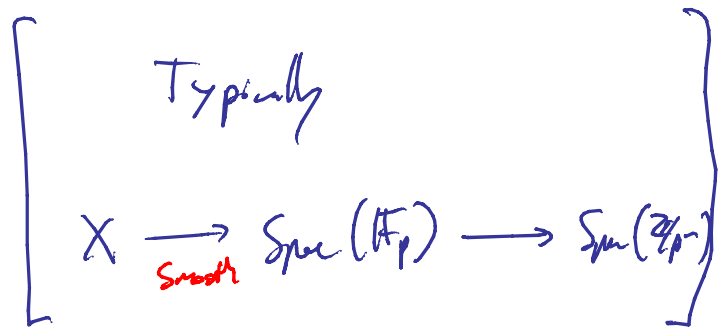
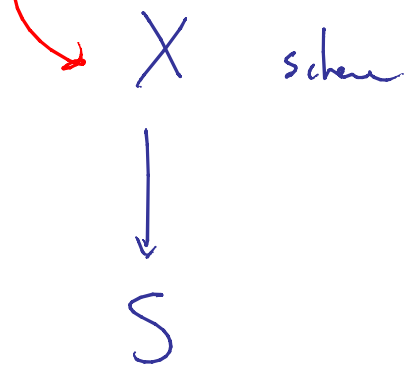
But! If we are thirsty for blood!

# CRYSTALLINE SITE

$(S, I, \gamma) =$  base P.D. scheme

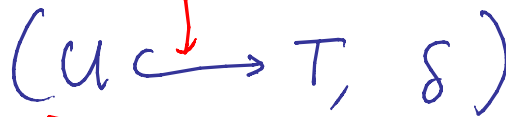
Assume  $\exists$  extensions  
of P.D. scheme  $\rightarrow X$

(commonly equal to  $(\mathbb{Z}/p^n, (p), \text{canonical})$ )



$\text{Cris}(X/S) =$  site consisting of

Objects:



closed  $S$ -immersion, nil test  
 $T \rightarrow S$  comp. w/ P.D. structure

Zariski  
open in  $X$

P.D. structure  
on ideal of  
closed immersion

Morphisms:

squares which commute,  
induce

Covdys:  $(U_i \hookrightarrow T_i, d_i)$

↓

$(U \hookrightarrow T, d)$

If  $U_i$  cover  $U$  Zariski-wise

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Can talk about sheaves on this  
Site.

$\mathcal{F} \in \text{Sh}(\text{Cris}(X/S), \text{Ab})$

Assum  $S = \text{affine}$

$R^* \Gamma \mathcal{F}$

What is  $\Gamma$ ?

don't have  
final object!

Local <sup>error</sup>  
Site  $\rightsquigarrow$  Topos

$$e \quad \text{Sh}(e, \text{Sets})$$

$$\Gamma \mathcal{F} = \text{Morph}(e, \mathcal{F})$$

$e =$  final object of topos

$=$  constant sheaf wh  
singleton output

Remark:  $\Gamma$  is difficult to compute  
in  $\text{Sh}(\text{Cris}(X/S))$

But if you have a good test  
pro-object:

e.g.

$$\text{Spec}(\mathbb{F}_p[x]) \rightarrow \text{Spec}(\mathbb{Z}/p^n)$$

Good test object:  $\mathbb{Z}/p^n[x]$

## De Rham - Witt complex:

Provides a suitable test object.

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Need Functoriality:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Would like morphism of sites:

giving  $g^* : \text{Sh}(\text{Crs}(X/S)) \longrightarrow \text{Sh}(\text{Crs}(X'/S'))$

would be  $g_{\text{Crs}}^* : \text{Crs}(X'/S') \longrightarrow \text{Crs}(X/S)$

**DOES NOT EXIST**

Why do we think we can fix this?

Groth knew something:

Comes out of work of Dwork  
(Minsky-Walshitzer)

$X$   $k$  perfect,  $\text{char} = p$   
 $\downarrow f$   
 $\text{Spec}(k)$

$X_1$   $X_2$   
 $\searrow f_1 \quad \swarrow f_2$  two lifts  
 $W(k)$

$\exists$  canonical iso:

$$H_{\text{dR}}^*(X_1) \cong H_{\text{dR}}^*(X_2)$$

$$X_i = (X_i)_F \quad F = W(k) \otimes \mathbb{Q}$$

(Actually  $H_{\text{dR}}^*(X_1) \cong H_{\text{dR}}^*(X_2)$ )

Groth: constructed morphism directly on  
the topos.

$(g^*, g_*)$

[topos why  $g^*$  nice  $\implies$  get  $g_*$  as  
Ryher Kan  
extension]

Cohens:  $(U, T, \delta) \in \text{Cris}(X/S)$

Consider:  
Shaf  $\tilde{T} = \text{Hom}(-, (U, T, \delta))$

$(g^* \tilde{T}) : (U', T', \delta') \mapsto$

P.D. Hom $_g((U', T', \delta'), (U, T, \delta))$

$$\text{P.D. Hom}_g((u', T, \delta'), (u, T, \delta))$$

$$:= \begin{cases} \emptyset, & g(u') \notin U \\ \{h: (T', \delta') \rightarrow (T, \delta)\} \end{cases}$$

$$\begin{array}{ccc} u' & \longrightarrow & u \\ \downarrow \delta' & & \downarrow \delta \\ T' & \xrightarrow{h} & T \\ \downarrow \delta' & & \downarrow \delta \\ S & \xrightarrow{\delta} & S \end{array}$$

$\text{Cris}(X/S)$  has a special object

Structure Sheaf:  $\mathcal{O}_{(X/S)_{\text{cris}}}$

$$(u, T, \delta) \longrightarrow \Gamma \mathcal{O}_T$$

Crystalline Cohomology:  $H_{\text{cris}}^*(X/S) := R^* \Gamma \mathcal{O}_{(X/S)_{\text{cris}}}$



