Kedlaya: Concrete Crystalline Cohomology

Note Title 3/5/2009

$$\begin{array}{l} X = \text{smooth over } \operatorname{Spec}(k) \\ \quad \uparrow \text{e.g.} \\ \downarrow \\ S = \operatorname{Spec}(W_n(k)) \\ \quad \uparrow \\ \quad \text{e.g.} \end{array}$$

Study "ways to thicken X to S"

$$\operatorname{Crys}(X/S) = \left\{ \begin{array}{l} (U, T, \delta) \\ U = \text{Zariski open in } X \\ T = \text{nilpotent thickening of } U/S \\ \delta = \text{divided power structure on} \\ \qquad \mathcal{I}(U \hookrightarrow T) \end{array} \right.$$

Structure sheaf

$$\mathcal{O}_{X/S,\mathrm{crys}} : (U, T, \delta) \longmapsto \mathcal{O}_T$$

Crystalline coh of sheaf $\mathcal{F}$ of $\mathcal{O}_{X/S,\mathrm{crys}}$-modules is the right derived functor of global sections

Global section = compatible family of sections over All Objects

$$H^*_{crys}(X/S) = R^*\Gamma\, \mathcal{O}_{X/S,\,crys}$$

Would prefer to compute a particular thickening plus extra data which remembers thickenings you didn't write down.

Crystal special kind of sheaf of $\mathcal{O}_{X/S,\,crys}$-modules

- rigid
- prolongs (already true of sheaf on crystalline site)

A <u>crystal</u> on (X/S) is a sheaf

of $\mathcal{O}_{X/S, \text{cris}}$ - modules

<u>s.t.</u> $\quad u : (U', T', \delta') \longrightarrow (U, T, \delta)$

for any

$$u^* \mathcal{F}_{(U,T,\delta)} \longrightarrow \mathcal{F}_{(U',T',\delta')}$$

$$\wr\wr$$

$$\mathcal{O}_{T'} \otimes_{\mathcal{O}_T} \mathcal{F}_{(U,T,\delta)}$$

is an isomorphism of $\mathcal{O}_T$ - modules

<u>Connections</u>: X/S smooth

$\mathcal{M}$ = quasi-coherent $\mathcal{O}_X$ - module

A <u>connection</u> on $\mathcal{M}$ (w.r.t S) is a morph

$$\nabla : \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

of sheaves of $\mathcal{O}_S$-modules

s.t.

$$\nabla(fm) = f\nabla(m) + m \otimes df$$

for f = section of $\mathcal{O}_X$
m = section of $\mathcal{M}$

We say ∇ is <u>integrable</u> or <u>flat</u>

if

$$\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \xrightarrow{\nabla^{(1)}} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^2_{X/S}$$

$$m \otimes df \longmapsto \nabla(m) \wedge df$$

composes to zero

<u>Consequence</u>

$$0 \to \mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega^1_{X/S} \xrightarrow{\nabla^{(1)}} \mathcal{M} \otimes \Omega^2_{X/S} \xrightarrow{\nabla^{(2)}} \cdots\cdots$$

is a complex of sheaves;

$$\mathcal{M} \otimes_{\mathcal{O}_{X,\nabla}} \Omega^\bullet_{X/S}$$

Can form:

$$\mathbb{H}^\bullet(\mathcal{M} \otimes_{\mathcal{O}_{X,\nabla}} \Omega^\bullet_{X/S})$$

Algebraic de Rham cohomology of X etc.

graded $\mathcal{O}_S$-module

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e.g. X affine: no higher coherent cohomology

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e.g. $X = \mathbb{P}^1_{\mathbb{Q}}$ $\qquad \mathcal{M} = \mathcal{O}_X$

$S = \mathrm{Spec}(\mathbb{Q})$ $\qquad \nabla = d$

$$\mathbb{P}^1_{\mathbb{Q}} = U_0 \cup U_\infty$$

$$U_i \cong \mathbb{A}^1_{\mathbb{Q}}$$

$$\begin{tikzcd}
\mathbb{Q}[x] \oplus \mathbb{Q}[x^{-1}] \arrow[r] \arrow[d, "d"] & \mathbb{Q}[x, x^{-1}] \arrow[d] \\
\mathbb{Q}[x]\{dx\} \oplus \mathbb{Q}[x^{-1}]\{dx^{-1}\} \arrow[r] & \mathbb{Q}[x, x^{-1}]\{dx\}
\end{tikzcd}$$

$= \text{"}-x^{-2}dx\text{"}$

<u>Calculation</u>!

$$H^0_{dR} = \mathbb{Q}$$

$$H^1_{dR} = 0 \quad \longleftarrow \text{interesting computation}$$

$$H^2_{dR} = \mathbb{Q}\left\{\frac{dx}{x}\right\}$$

X/S smooth

$$\begin{array}{ccc} X \times_S X & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \\ X & & \end{array}$$

$X \overset{\Delta}{\hookrightarrow} X \times_S X$ closed immersion

(smooth $\Rightarrow$ separated)

$$\tilde{I} = \tilde{I}(X \hookrightarrow X \times_S X)$$

$$P^n = \mathcal{O}_{X \times_S X} / \tilde{I}^{n+1}$$

Note:

$$0 \to \Delta_* \Omega^1_{X/S} \to P^1 \to \Delta_* \mathcal{O}_X \to 0$$

Thm A quasi-coh connection on an $\mathcal{O}_X$-module M is specified by an isomorphism

$$\theta : \pi_1^* M|_{P_1} \cong \pi_2^* M|_{P^1}$$

agreeing on P_0

i.e. $X = \operatorname{Spec} R$ $\quad M/R$

$S = \operatorname{Spec} k$

$$\Theta : M \otimes_{\pi_1} P^1 \cong M \otimes_{\pi_2} P^1$$

induces identity mod Ω^1

$$\nabla(m) = \Theta(m \otimes 1) - m \otimes 1 \in M \otimes_R \Omega^1$$

An integrable connection is a connection for which

Θ satisfies a cocycle condition

on $X \times_S X \times_S X$

(cf Dix Exposés)

Infinitesimal site

X/S, char 0 case

Crystal $\longleftrightarrow$ $\mathcal{O}_X$-module + integrable connection

Rigidity condition

$\Rightarrow$ can reconstruct the values of crystal using connection to get values on inf nbhds of $X \overset{\Delta}{\hookrightarrow} X \times X$

Cohomology $=$ de Rham Cohomology

Smooth char 0 $\Rightarrow$ value on p' determines value on p^n

(Taylor series map)

$\sum \frac{f^{(n)}(0)}{n!} x^n$

have to divide by some stuff

In crystalline site case

(c.f. Katz: Slope filtration of F-crystals)

$X/_k$ $\quad k$ - perfect char p

$S = W_n(k)$

$$\begin{pmatrix} \text{Crystal on} \\ X/_S \end{pmatrix} \rightsquigarrow \begin{pmatrix} \text{for } \tilde{X} = \underline{\text{some}} \text{ smooth lift of } X/_S \\ \text{get } \mathcal{O}_{\tilde{X}}\text{-module on } \tilde{X} \\ + \text{ integrable connection} \end{pmatrix}$$

If $\tilde{\tilde{X}}$ = smooth formal lift $/ \operatorname{Spf}(W(k))$

get $\mathcal{O}_{\tilde{\tilde{X}}}$-modules, + integrable connection

$$H_{cris}(X/_S) \cong H_{dR}(\tilde{\tilde{X}}/_S)$$

e.g.

$$\mathbb{P}^1_{\mathbb{F}_p}$$

$$H^*_{crys}(\mathbb{P}^1_{\mathbb{F}_p}/\mathbb{Z}_p) \cong H^*_{dR}(\mathbb{P}^1_{\mathbb{Z}_p}/\mathbb{Z}_p)$$

$$= \begin{cases} \mathbb{Z}_p & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}_p & * = 2 \end{cases}$$

(Warning: $H_{dR}(\mathbb{A}^1_{\mathbb{Z}_p})$ has lots of p-torsion

but gets cancelled when you glue)

Given lift $\tilde{\tilde{X}}$

+ $\mathcal{O}_{\tilde{\tilde{X}}}$-module $\rightsquigarrow$ crystal

w/ connection

+ criterion "convertible"

X smooth $\Rightarrow$ obstructions to lifting vanish locally

X smooth affine $\Rightarrow$ always can lift

X = curve $\Rightarrow$ always have lift

X = surface $\Rightarrow$ cannot always lift

Crystalline case

Need all P^n's universal equal char thickenings

+ all $W_n(k)$'s universal unequal char thickenings

$\rightsquigarrow$ crystal

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de Rham-Witt Complex

$$\begin{tikzcd} R/k \arrow[r, mapsto] & W(R) \arrow[r, squiggly] & \Omega^\bullet_{W(R)/W(k)} \arrow[r, two heads] & W\Omega_{R/k} \arrow[loop above, "F,V"] \end{tikzcd}$$

k = perfect, char = p

get

$$\begin{tikzcd} \underset{\text{scheme}}{X} \arrow[r] & \underset{\text{sheaf}}{W(X)} \arrow[r, squiggly] & \Omega^\bullet_{W(X)/W(k)} \arrow[r, two heads] & W\Omega_{X/k} \arrow[loop below, "F,V"] \end{tikzcd}$$

Crystalline coh of X

$$H_{cris}(X) \rightsquigarrow H_{dR}(\tilde{X}) = \mathbb{H}^*(\Omega^\bullet_{\tilde{X}}) \xrightarrow[\cong]{} \mathbb{H}^*(W\Omega^\bullet_X)$$

smooth lift

this is the universal test pro-object in the crystalline site connection

formal completion of diagonal $\rightsquigarrow$ de Rham cx

? $\rightsquigarrow$ de Rham Witt cx

crystal

M = crystal $\rightsquigarrow$ module w/ de Rham–Witt connection.