Zacharevich - Intro to alg K-thy

Note Title 4/2/2009

$$K_0(R) = \frac{\mathbb{Z}\{\overbrace{\text{iso classes of f.g. proj mods}/R}^{\text{Proj}(R)}\}}{[M \oplus N] - [M] - [N]}$$

"how to put together free mods"

$$K_1(R) = GL(R)^{ab}$$

$$GL(R) = \operatorname{colim} GL_\infty(R)$$

"automorphisms of free mods"

R = Dedekind Domain

$$\bigoplus_{m} K_1(R/m) \longrightarrow K_1(R) \longrightarrow K_1(F) \longrightarrow \bigoplus_{m} K_0(R/m) \longrightarrow K_0(R) \longrightarrow K_0(F)$$

m max ideal R

F: fld of fractions

looks "topological"

$$K_0(R) = \frac{\mathbb{Z}[\mathrm{Proj}(R)]}{[M \oplus N] - [M] - [N]}$$

$$= \mathrm{Groth}(\mathrm{Proj}(R), \oplus)$$

$$= \mathrm{Groth}(\pi_0 \, B\mathrm{Proj}(R), \oplus)$$

nonphisms = isos

$(\mathrm{Proj}(R), \oplus)$ = sym monoidal groupoid

Want to "commute" π_0 and Groth

$$\overset{?}{=} \pi_0 \, \Omega B (B\mathrm{Proj}(R))$$

claim

π_0 = set

π_1 = Gp

$B\mathrm{Proj}(R)$ = top'l monoid

<u>Thm</u>: Gp completion thm:

Let M be a top'l monoid

If $\pi_0 M$ is in center of

$H_*(M)$, then

$$H_*(M)[\pi_0(M)^{-1}] \xrightarrow[\cong]{} H_*(\Omega BM)$$

$$M \rightsquigarrow \Omega BM$$

- "doesn't change homology too much"
- "does change π_*"

<u>Def</u>: The gp completion of a top'l monoid X is a space Y

$$X \xrightarrow[f]{} Y$$

Such that

(a) $\pi_0 X \longrightarrow \pi_0(Y)$ is a gp completion

(b) $H_*(X)[\pi_0 X^{-1}] \longrightarrow H_*(Y)$

is an isomorphism

Def: $K_i(R) := K_i\, \Omega B\, B\mathrm{Proj}(R)$

$\mathcal{C}$ = cat f.g. proj modules/R

morph = all homomorphisms

$M' \longrightarrow M \longrightarrow M''$ SES.

Consider new cat $Q\mathcal{C}$

Slogan: Ob: Same as $\mathcal{C}$

morphisms: $M \longrightarrow N$ if

M "sits inside" N

A morphism $M \longrightarrow N$ in QC are isomorphism classes of diagrams

$$\begin{tikzcd} & X \arrow[dl, two heads] \arrow[dr, hook] & \\ M & & N \end{tikzcd}$$

"admissible"
if ker/coker
exist
in $\mathrm{Proj}(R)$
[ker automatic]

How do we compose these?

$$\begin{tikzcd} & & Z \arrow[dl, two heads] \arrow[dr, hook] & & \\ & X \arrow[dl, two heads] \arrow[dr, hook] & & Y \arrow[dl, two heads] \arrow[dr, hook] & \\ M & & N & & P \end{tikzcd}$$

pullback

Prop: (1) Any morphism can be factored into a map whose mono is id, followed by one whose epi is id.

$$\begin{tikzcd}
 & X \arrow[dl, two heads] \arrow[dr, equal] & & X \arrow[dl, equal] \arrow[dr, hook] & \\
M & & X & & N
\end{tikzcd}$$

(2) For any module M,

$\exists$ 2 morphisms in $\mathrm{Hom}_{QC}(0, M)$:

$$\begin{tikzcd}
 & M \arrow[dl, two heads] \arrow[dr, equal] & & & & 0 \arrow[dl, equal] \arrow[dr, hook] & \\
0 & & M & & 0 & & M
\end{tikzcd}$$

So we have a loop in BQC!

Thm: $\pi_1 BQC = K_0(R)$

Idea:

$$K_0(R) \longrightarrow \pi_1(BQ\mathcal{C})$$

$$M \longmapsto \begin{tikzcd} & 0 \arrow[dr, hook] & \\ 0 \arrow[ur, equal] & & M \arrow[dl, equal] \\ & M \arrow[ul, two heads] & \end{tikzcd} = \tilde{\iota}_M$$

loop

Well defined:

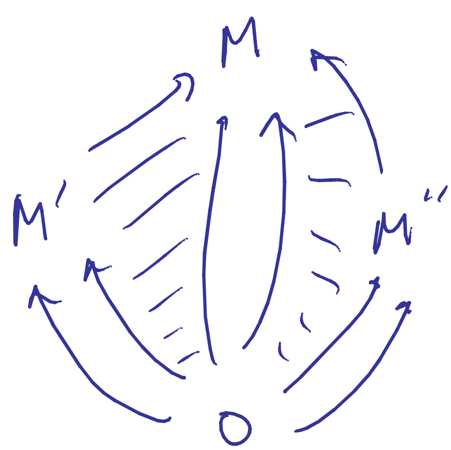
$$M' \longrightarrow M' \oplus M'' \longrightarrow M''$$

$$M'' \longrightarrow M' \oplus M'' \longrightarrow M'$$

Need: for $M' \rightarrowtail M \twoheadrightarrow M''$

need $[\tilde{\iota}_M] = [\tilde{\iota}_{M'}][\tilde{\iota}_{M''}]$ (*)

In QC:



get sphere w/ 3 holes making (*) hold

new def:

$$K_i(R) = \pi_{i+1}(BQ\,\mathcal{C})$$

Q: does this agree w/ old def?

Generalizations:

Claim: If S = symmetric monoidal groupoid

$$K_i(S) = \pi_i\, \Omega BBS$$

"works"

Claim: We used
- exact sequence
- pullbacks

$\mathcal{C}$ is exact if it is a subcat of an abelian cat, closed under extensions

If $\mathcal{C}$ is exact:

$$K_i(\mathcal{C}) := \pi_{i+1}(B \mathbb{Q} \mathcal{C})$$ "works"

<u>Thm:</u> If $S = \text{iso}\ \mathcal{C}$

and $\mathcal{C}$ is split exact

$$\Rightarrow K_i(\mathcal{C}) = K_i(S)$$

— Break —

We will give a third "gp completion"

Looks like "field of fractions"

<u>Categorical</u> <u>versions</u> S = sym monoidal space

<u>Def:</u> S acts on $\mathcal{D}$ if

there is a functor

$$S \times \mathcal{D} \longrightarrow \mathcal{D}$$

$$
\begin{tikzcd}
S \times S \times S \arrow[r] \arrow[d] & S \times \mathcal{D} \arrow[d] \\
S \times \mathcal{D} \arrow[r] & \mathcal{D}
\end{tikzcd}
\quad , \qquad
\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[dr, equal] & S \times \mathcal{D} \arrow[d] \\
& \mathcal{D}
\end{tikzcd}
$$

<u>Example:</u>

S acts on S

$S^{-1}\mathcal{D}$ = another cat

Objects: (s, d)

morphisms: $(s_1, d_1) \longrightarrow (s_2, d_2)$

triples (r, f, g) (iso classes of such $r \cong r'$)

$r \in S$ $\qquad f : r \oplus s_1 \longrightarrow s_2$

$\qquad\qquad g : r \oplus d_1 \longrightarrow d_2$

Thm: If $\forall\ s \in S$

$$\mathrm{Aut}(d) \longrightarrow \mathrm{Aut}(s \oplus d)$$

is injective. (and $\mathcal{D}$ is a groupoid)

Then: $\mathcal{D} \longrightarrow S^{-1}\mathcal{D}$

induces:

$$H_*(B\mathcal{D})[\pi_0(BS)^{-1}] \xrightarrow{\cong} H_*(BS^{-1}\mathcal{D})$$

In particular, if $S = \mathcal{D}$

then $BS \longrightarrow BS^{-1}S$

is a gp completion.

Need: $\pi_i(BS^{-1}S) \cong \pi_{i+1}(BQ\mathcal{C})$

will do this by: fibn $S^{-1}\mathcal{E}\mathcal{C}$

$$
\begin{tikzcd}
BS^{-1}S \arrow[r] & BS^{-1}\mathcal{E}\mathcal{C} \simeq * \arrow[d] \\
 & BQ\mathcal{C}
\end{tikzcd}
$$

w/ LES in htpy

$\mathcal{E}\mathcal{C}$: <u>objects:</u> exact sequences in $\mathcal{C}$

<u>morphisms:</u>

$$
\begin{tikzcd}
A \arrow[r, hook] & B \arrow[r, two heads] & C \\
A' \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & B \arrow[u, equal] \arrow[r, two heads] \arrow[d, hook] & C'' \arrow[u] \arrow[d, hook] \\
A' \arrow[r, hook] & B' \arrow[r, two heads] \arrow[ur, phantom, "\lrcorner"] & C'
\end{tikzcd}
$$

Have a funct

$$\Sigma\mathcal{C} \longrightarrow Q\mathcal{C}$$

$$A \hookrightarrow B \twoheadrightarrow C \longmapsto C$$

S acts on $\Sigma\mathcal{C}$

by
$$\begin{array}{c} A \hookrightarrow B \twoheadrightarrow C \\ \downarrow \\ S \oplus A \hookrightarrow S \oplus B \twoheadrightarrow C \end{array}$$

<u>Actually</u>: get $\hat{T} : S^{-1}\Sigma\mathcal{C} \to Q\mathcal{C}$

<u>Claim</u>: $S^{-1}\Sigma\mathcal{C}$ contractible

Suffices to prove $\Sigma\mathcal{C}$ is contractible

Thm: Quillen's Thm A

$F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor

s.t. $\forall \ B \in \mathcal{B}$

$B \downarrow F$ is contractible

(or $F \downarrow B$)

$\Longrightarrow$ $B\mathcal{A} \longrightarrow B\mathcal{B}$

is an equivalence

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look at:

same objects
only monomorphisms

$$m : \mathcal{E}\mathcal{C} \longrightarrow \text{mono } Q\mathcal{C}$$

$$\begin{array}{ccc} \cup & & \cup \\ A \hookrightarrow B \twoheadrightarrow C & \longmapsto & B \end{array}$$

$0 \in Q\mathcal{C}$ initial $\Longrightarrow$ $Q\mathcal{C}$ contractible look at $(m \downarrow B)$

objects: $(A \to B' \to C\ ,\ \ B' \hookrightarrow B)$

any such object has a map to $0 \hookrightarrow B = B$

$$
\begin{tikzcd}
A \arrow[r, hook] & B' \arrow[r] & C \\
0 \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & B' \arrow[u, equal] \arrow[r, equal] \arrow[d, hook] & B' \arrow[u] \arrow[d, hook] \\
0 \arrow[r, hook] & B \arrow[r, equal] & B
\end{tikzcd}
$$

terminal object

$\Longrightarrow \quad (m \downarrow B) \simeq *$

Want: LES in homotopy:

$$BS^{-1}S \longrightarrow BS^{-1}\mathcal{E}\mathcal{C} \longrightarrow BQ\mathcal{C}$$

Need:

Quillen's thm B

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

all $f : X \to Y \in \mathcal{B}$ induce

$$(Y \downarrow F) \xrightarrow[f^*]{} (X \downarrow F)$$

If all f^* are htpy equivalences

get: LES

$$\cdots \longrightarrow \pi_i(B \downarrow F, A) \longrightarrow \pi_i(B\mathcal{A}, A) \longrightarrow \pi_i(B\mathcal{B}, B) \longrightarrow \cdots$$

<u>Choose:</u>

(for simplicity)

$A: \ (0, 0 = 0 = 0) \in S^{-1}\mathcal{E}\mathcal{C}$

$B = 0 \quad \in Q\mathcal{C}$

$$B(C \downarrow \hat{T}) \simeq B(\hat{T}^{-1}(C))$$

since $\hat{T}^{-1}(C) \hookrightarrow (C \downarrow \hat{T})$ has a right adjoint

$$\hat{T}^{-1}(0) = S^{-1}S$$

<u>So:</u>

$$B(0 \downarrow \hat{T}) \simeq BS^{-1}S$$

$$S \longrightarrow \mathcal{E}\mathcal{C} \qquad \forall \ c \in \mathcal{C}$$

$$S \longrightarrow S \oplus C \longrightarrow C$$

gives $\quad BS^{-1}S \simeq B\hat{T}^{-1}(C) \ \forall C$

(here is where you
need to use
the fact that
exact sequences
split.

$f \in QC$ factors by

$mono \circ epi$