A C_2-EQUIVARIANT ANALOG OF MAHOWALD'S THOM SPECTRUM THEOREM

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ABSTRACT. We prove that the C_2-equivariant Eilenberg-MacLane spectrum associated with the constant Mackey functor $\underline{\mathbb{F}}_2$ is equivalent to a Thom spectrum over $\Omega^\rho S^{\rho+1}$.

1. INTRODUCTION

Let μ be the Möbius bundle over S^1, regarded as a virtual bundle of dimension 0. The mod 2 Moore spectrum is the Thom spectrum

$$M(2) \simeq (S^1)^\mu.$$

The classifying map for μ extends to a double loop map

$$\widetilde{\mu} : \Omega^2 S^3 \to BO.$$

Mahowald proved the following theorem [Mah77]:

Theorem 1.1 (Mahowald). *There is an equivalence of spectra*

$$(\Omega^2 S^3)^{\widetilde{\mu}} \simeq H\mathbb{F}_2.$$

The bundle μ may also be regarded as a C_2-equivariant virtual bundle over S^1, by endowing both S^1 and the bundle with the trivial action. Since $B_{C_2}O$ is an equivariant infinite loop space [Ati68], the classifying map for μ extends to an Ω^ρ-map

$$\widetilde{\mu} : \Omega^\rho S^{\rho+1} \to B_{C_2}O.$$

Here, ρ is the regular representation of C_2. The purpose of this paper is to prove the following.

Theorem 1.2. *There is an equivalence of C_2-spectra*

$$(\Omega^\rho S^{\rho+1})^{\widetilde{\mu}} \simeq H\underline{\mathbb{F}}_2.$$

(Here, $\underline{\mathbb{F}}_2$ denotes the constant Mackey functor with value $\mathbb{F}_2$.)

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Conventions. Equivariant objects in this paper either live in Top^{C_2}, the category of C_2-spaces, or Sp^{C_2}, the category of genuine C_2-spectra. In both of these categories, the equivalences are those equivariant maps which induce equivalences on both the C_2-fixed points spectrum and the underlying spectrum. We let $\mathbf{H}$ denote the Eilenberg-Maclane spectrum $H\underline{\mathbb{F}}_2$, with underlying spectrum $H := H\mathbb{F}_2$. We use $\mathbf{H}_\star$ and $\pi_\star^{C_2}$ to denote $RO(C_2)$-graded homology and homotopy *groups* (i.e. *not* the Mackey functors) of C_2-equivariant spaces and spectra, and H_* and π_* to denote the ordinary homology and homotopy groups of non-equivariant spaces and spectra. We let σ denote the sign representation of C_2, and let $\rho = 1 + \sigma$ denote the regular representation. For a representation V, $S(V)$ denotes the unit sphere in V, and S^V denotes its one point compactification, and $|V|$ denotes its dimension.

2. Equivariant preliminaries

Euler class. Let a denote the Euler class in $\pi_{-\sigma}^{C_2} S$, given geometrically by the inclusion

$$S^0 \hookrightarrow S^\sigma.$$

There is a cofiber sequence

$$C_{2+} \to S^0 \hookrightarrow S^\sigma \tag{2.1}$$

so the cofiber of a is stably given by

$$Ca \simeq \Sigma^{1-\sigma} C_{2+}. \tag{2.2}$$

The equivalence of underlying spectra

$$(S^1)^e \simeq (S^\sigma)^e \tag{2.3}$$

induces an equivalence of C_2-spectra

$$C_{2+} \wedge S^1 \simeq C_{2+} \wedge S^\sigma.$$

Therefore, the equivalence (2.2) can actually be regarded as giving an equivalence

$$Ca \simeq C_{2+}.$$

It follows that Ca is a commutative ring spectrum. The adjoint of the equivalence (2.3) gives a C_2-equivariant map

$$C_{2+} \wedge S^1 \to S^\sigma$$

which, by the self-duality of C_{2+}, gives a map

$$u : S^1 \to C_{2+} \wedge S^\sigma \simeq Ca \wedge S^\sigma$$

which serves as a Thom class for the representation σ. For $X \in \mathrm{Sp}^{C_2}$, we have

$$\pi_k^{C_2}(X) \cong \pi_k(X^{C_2}),$$
$$\pi_V^{C_2}(X \wedge Ca) \cong \pi_{|V|}(X^e).$$

Said differently,

$$\pi_\star^{C_2}(X \wedge Ca) \cong \pi_*(X^e)[u^\pm]. \tag{2.4}$$

Tate square. We will let

$$X^h := F(EC_{2+}, X),$$
$$X^\Phi := X \wedge \widetilde{EC}_2$$

denote the homotopy completion and geometric localization of X, respectively. The fixed points of X^h are the homotopy fixed points of X, and the fixed points of X^Φ are the geometric fixed points of X. X is recovered from these approximations by the pullback ("Tate square") [GM95]

$$\begin{array}{ccc} X & \longrightarrow & X^\Phi \\ \downarrow & & \downarrow \\ X^h & \longrightarrow & X^t \end{array}$$

where the spectrum X^t is the equivariant Tate spectrum

$$X^t := (X^h)^\Phi.$$

Note that a generalization of the argument establishing (2.2) yields an equivalence

$$\Sigma^{k\sigma-1}C(a^k) \simeq S(k\sigma)_+.$$

Taking a colimit, we see that we have

$$\operatorname*{hocolim}_k \Sigma^{k\sigma-1}C(a^k) \simeq EC_{2+},$$
$$\operatorname*{hocolim}_k S^{k\sigma} \simeq \widetilde{EC}_2.$$

It follows that homotopy completion and geometric localization can be reinterpreted as a-completion and a-localization:

$$X^h \simeq X_a^\wedge,$$
$$X^\Phi \simeq X[a^{-1}].$$

In this manner, the Tate square is equivalent to the "a-arithmetic square"

$$\begin{array}{ccc} X & \longrightarrow & X[a^{-1}] \\ \downarrow & & \downarrow \\ X_a^\wedge & \longrightarrow & X_a^\wedge[a^{-1}] \end{array}$$

Using (2.4), the a-Bockstein spectral sequence takes the form

$$E_1^{*,*} = \pi_*(X^e)[u^\pm, a] \Rightarrow \pi_\star^{C_2}(X^h).$$

The a-Bockstein spectral sequence can be regarded as an $RO(C_2)$-graded version of the homotopy fixed point spectral sequence (see [HM17, Lem. 4.8]).

The mod 2 Eilenberg-MacLane spectrum. We have [HK01]

$$\pi_\star^{C_2}\mathbf{H} = \mathbb{F}_2[a,u] \oplus \frac{\mathbb{F}_2[a,u]}{(a^\infty, u^\infty)}\{\theta\}$$

where

$$|u| = 1-\sigma,$$
$$|\theta| = 2\sigma - 2.$$

The a-u divisible factor in $\pi_\star \mathbf{H}$ is best understood from the Tate square, using

$$\pi_\star^{C_2}\mathbf{H}^h \cong \mathbb{F}_2[a, u^{\pm 1}],$$
$$\pi_\star^{C_2}\mathbf{H}^\Phi \cong \mathbb{F}_2[a^{\pm 1}, u].$$

Actually, the second isomorphism lifts to an equivalence

$$\mathbf{H}^{\Phi C_2} \simeq H[a^{-1}u] := \bigvee_{i \geqslant 0} \Sigma^i H$$

so we have

$$\mathbf{H}^\Phi_\star X \cong H_*(X^{\Phi C_2})[a^{\pm 1}, u]$$

and, restricting the grading to trivial representations, we get

$$\mathbf{H}^\Phi_* X \cong H_*(X^{\Phi C_2})[a^{-1}u]. \tag{2.5}$$

By applying $\pi_V^{C_2}$ to the map

$$\mathbf{H} \wedge X \to \mathbf{H} \wedge X \wedge Ca$$

we get a homomorphism

$$\Phi^e : \mathbf{H}_V(X) \to H_{|V|}(X^e). \tag{2.6}$$

Taking geometric fixed points of a map

$$S^V \to \mathbf{H} \wedge X$$

gives a map

$$S^{V^{C_2}} \to \mathbf{H}^{\Phi C_2} \wedge X^{\Phi C_2}$$

Using (2.5) and passing to the quotient by the ideal generated by $a^{-1}u$, we get a homomorphism

$$\Phi^{C_2} : \mathbf{H}_V(X) \to H_{|V^{C_2}|}(X^{\Phi C_2}). \tag{2.7}$$

A useful lemma. Our main computational lemma is the following.

Lemma 2.8. *Suppose that $X \in \mathrm{Sp}^{C_2}$ and suppose that $\{b_i\}$ is a set of elements of $\mathbf{H}_\star(X)$ such that*

1. *$\{\Phi^e(b_i)\}$ is a basis of $H_*(X^e)$, and*
2. *$\{\Phi^{C_2}(b_i)\}$ is a basis of $H_*(X^{\Phi C_2})$.*

Then $\mathbf{H}_\star(X)$ is free over $\mathbf{H}_\star$, and $\{b_i\}$ is a basis.

Proof. The set $\{b_i\}$ corresponds to a map

$$\mathbf{H} \wedge \bigvee S^{|b_i|} \to \mathbf{H} \wedge X.$$

Assumption (1) implies this map is an equivalence upon applying Φ^e, while assumption (2) implies this map is an equivalence upon applying Φ^{C_2}. The result follows. □

3. Homology of ρ-loop spaces

We spell out some specific algebraic structure carried by the equivariant homology of a ρ-loop space. A more detailed and general study of this algebraic structure can be found in [Hil17].

Products. Suppose $X = \Omega^\rho Y \in \mathrm{Top}^{C_2}$ is a ρ-loop space. Then X is in particular a 1-loop space, and is therefore an equivariant H-space with product

$$m : X \times X \to X.$$

However, the σ-loop space structure also endows X with a twisted product related to the transfer. Namely, let

$$S^\sigma \to S^\sigma/S^0 \approx C_{2+} \wedge S^1$$

be the pinch map. This gives rise to a twisted product

$$\widetilde{m} : N^\times \Omega Y \to \Omega^\sigma Y$$

where

$$N^\times Z := \mathrm{Map}(C_2, Z) = Z \underset{\overleftrightarrow{C_2}}{\times} Z$$

is the norm with respect to Cartesian product (i.e. the coinduced space). In particular, there is a map

$$\widetilde{m} : N^\times \Omega^2 Y \to X. \tag{3.1}$$

Upon applying fixed points to the map (3.1), we get an additive transfer

$$t : X^e \to X^{C_2}. \tag{3.2}$$

In homology, the H-space structure give rise to a product

$$m : \mathbf{H}_V X \otimes \mathbf{H}_W X \to \mathbf{H}_{V+W} X.$$

Using the equivariant commutative ring spectrum structure of $\mathbf{H}$ [Ull13], the twisted product $\widetilde{m}$ gives rise to a "norm map"(see [BH15, Thm. 7.2])

$$n : H_k X^e \to \mathbf{H}_{k\rho} X.$$

Dyer-Lashof operations. X has even more structure: X is an E_ρ-algebra [GM17]. Specifically, regard $S(\rho)$ as a $C_2 \times \Sigma_2$-space where C_2 acts on ρ and Σ_2 acts antipodally. Then the E_ρ-structure gives a map

$$S(\rho) \times_{\Sigma_2} X^{\times 2} \to X.$$

Note that $\mathbf{H}$ is itself an E_ρ-ring spectrum, because it is actually an equivariant commutative ring spectrum, so $\mathbf{H} \wedge X_+$ is an E_ρ-ring in $\mathbf{H}$-modules. Given $x \in \mathbf{H}_V(X)$, represented by a map

$$x : S^V \to \mathbf{H} \wedge X_+,$$

there is an induced composite

$$\begin{aligned} \mathbf{H} \wedge S(\rho)_+ \wedge_{\Sigma_2} S^{2V} \xrightarrow{1\wedge 1\wedge x\wedge x} & \mathbf{H} \wedge S(\rho)_+ \wedge_{\Sigma_2} (\mathbf{H} \wedge X_+)^{\wedge 2} \\ \to & \mathbf{H} \wedge \mathbf{H} \wedge X_+ \\ \to & \mathbf{H} \wedge X_+ \end{aligned}$$

(where the unlabeled maps come from the E_ρ-ring and $\mathbf{H}$-module structure of $\mathbf{H} \wedge X_+$). Applying $\pi_\star^{C_2}$, we get a total power operation

$$\mathcal{P}(x) : \widetilde{\mathbf{H}}_\star(S(\rho)_+ \wedge_{\Sigma_2} S^{2V}) \to \mathbf{H}_\star X.$$

For the purposes of this paper we will be only concerned with the case of $V = k\rho - \sigma$ for $k \in \mathbb{Z}$.

Proposition 3.3. *We have*

$$\widetilde{\mathbf{H}}_\star \left(S(\rho)_+ \wedge_{\Sigma_2} S^{2(k\rho-\sigma)}\right) \cong \mathbf{H}_\star\{e_{2k\rho-\sigma-1}, e_{2k\rho-\sigma}\}.$$

Proof. Consider the following cofiber sequences:

$$S^{2(k-1)\rho} \to S(\rho)_+ \wedge_{\Sigma_2} S^{2((k-1)\rho)} \to S^{2(k-1)\rho+\sigma} \tag{3.4}$$

$$\Sigma S(\rho)_+ \wedge_{\Sigma_2} S^{2((k-1)\rho)} \to S(\rho)_+ \wedge_{\Sigma_2} S^{2(k\rho-\sigma)} \to \Sigma^{2(k\rho-\sigma)} S(\rho)_+ \tag{3.5}$$

The sequence (3.4) arises from Theorem 2.15 of [Wil17] and the second arises from the $(C_2 \times \Sigma_2)$-equivariant inclusion $\Sigma S^{2((k-1)\rho)} \to S^{2(k\rho-\sigma)}$, where both C_2 and Σ_2 act trivially on the first suspension coordinate.

In (3.4), the boundary map on $\mathbf{H}_\star$ is zero because the group

$$\left[S^{2(k-1)\rho+\sigma}, \Sigma^{2(k-1)\rho+1}\mathbf{H}\right] = \mathbf{H}_{-1+\sigma}$$

is zero. Thus

$$\widetilde{\mathbf{H}}_\star \left(S(\rho)_+ \wedge_{\Sigma_2} S^{2(k-1)\rho}\right) \cong \mathbf{H}_\star\{e_{2(k-1)\rho}, e_{2(k-1)\rho+\sigma}\} = \mathbf{H}_\star\{e_{2k\rho-2\sigma-2}, e_{2k\rho-\sigma-2}\}.$$

Now we turn to the second cofiber sequence. Notice that $S(\rho)_+$ is C_2-equivariantly equivalent to $S^\sigma \vee S^0$. From the previous computation, the boundary is then determined by elements in the following four groups:

$$\begin{aligned}
\partial_1 &\in \left[S^{2(k\rho-\sigma)}, \Sigma^{2k\rho-2\sigma}\mathbf{H}\right] = \mathbf{H}_0 \\
\partial_2 &\in \left[S^{2(k\rho-\sigma)+\sigma}, \Sigma^{2k\rho-\sigma}\mathbf{H}\right] = \mathbf{H}_0 \\
\partial_3 &\in \left[S^{2(k\rho-\sigma)}, \Sigma^{2k\rho-\sigma}\mathbf{H}\right] = \mathbf{H}_{-\sigma} = 0 \\
\partial_4 &\in \left[S^{2(k\rho-\sigma)+\sigma}, \Sigma^{2k\rho-2\sigma}\mathbf{H}\right] = \mathbf{H}_{\sigma} = 0
\end{aligned}$$

Elements of $\mathbf{H}_0$ are determined by their restriction to H_0, and comparison with the underlying homology forces $\partial_1 = 1$ and $\partial_2 = 0$. The result follows. □

Thus we get a pair of Dyer-Lashof operations

$$\begin{aligned}
Q^{k\rho} &: \mathbf{H}_{k\rho-\sigma}X \to \mathbf{H}_{2k\rho-\sigma}X, \\
Q^{k\rho-1} &: \mathbf{H}_{k\rho-\sigma}X \to \mathbf{H}_{2k\rho-\sigma-1}X
\end{aligned}$$

given by the formulas

$$\begin{aligned}
Q^{k\rho}(x) &:= \mathcal{P}(x)(e_{2k\rho-\sigma}), \\
Q^{k\rho-1}(x) &:= \mathcal{P}(x)(e_{2k\rho-\sigma-1}).
\end{aligned}$$

Remark 3.6. If X is actually an equivariant infinite loop space, then $\mathbf{H}_\star X$ has an action by equivariant Dyer-Lashof operations [Wil17], and these operations agree with those defined in that paper.

Compatibility with fixed points. The compatibility of all this structure with the maps Φ^e and Φ^{C_2} of (2.6) and (2.7) is summarized as follows.

Products: Note that X^e is an E_2-algebra, and X^{C_2} is an E_1-algebra. The maps Φ^e and Φ^{C_2} are algebra homomorphisms.

Norms: The following diagram commutes:

$$\begin{array}{ccccc}
 & & H_k X^e & & \\
 & {\scriptstyle t}\swarrow & \downarrow {\scriptstyle n} & \searrow{\scriptstyle \mathrm{Fr}} & \\
H_k X^{C_2} & \xleftarrow{\Phi^{C_2}} & \mathbf{H}_{k\rho}X & \xrightarrow{\Phi^e} & H_{2k}X^e
\end{array}$$

Here t is the transfer (3.2) and Fr is the squaring map (Frobenius).

Dyer-Lashof operations: The following diagrams commute, where $\epsilon = 0, 1$:

$$\begin{array}{ccc}
\mathbf{H}_{k\rho-\sigma}X & \xrightarrow{\Phi^e} & H_{2k-1}X^e \\
{\scriptstyle Q^{k\rho-\epsilon}}\downarrow & & \downarrow{\scriptstyle Q^{2k-\epsilon}} \\
\mathbf{H}_{2k\rho-\sigma-\epsilon}X & \xrightarrow{\Phi^e} & H_{4k-1-\epsilon}X^e
\end{array}$$

$$\begin{array}{ccc} \mathbf{H}_{k\rho-\sigma}X & \xrightarrow{\Phi^{C_2}} & H_k X^{C_2} \\ {\scriptstyle Q^{k\rho}}\downarrow & & \downarrow{\scriptstyle \mathrm{Fr}} \\ \mathbf{H}_{2k\rho-\sigma}X & \xrightarrow[\Phi^{C_2}]{} & H_{2k} X^{C_2} \end{array}$$

4. Homology of $\Omega^\rho S^{\rho+1}$

Theorem 4.1. *There is an* additive *isomorphism (of $\mathbf{H}_\star$-modules)*

$$\mathbf{H}_\star \Omega^\rho S^{\rho+1} \cong \mathbf{H}_\star \otimes E[t_0, t_1, \ldots] \otimes P[e_1, e_2, \ldots]$$

with

$$|t_i| = 2^i\rho - \sigma,$$
$$|e_i| = (2^i - 1)\rho.$$

Proof. Note that we have

$$H_* \Omega^2 S^3 = \mathbb{F}_2[x_1, x_2, \ldots]$$

with

$$|x_i| = 2^i - 1.$$

Here x_1 is the fundamental class ι_1, and

$$x_i := Q^{2^i} Q^{2^{i-1}} \cdots Q^2 x_1.$$

Define $t_0 \in \mathbf{H}_1 \Omega^\rho S^{\rho+1}$ to be the fundamental class, and define the other "generators" e_i and t_i by

$$e_i := n(x_i),$$
$$t_i := Q^{2^i\rho} Q^{2^{i-1}\rho} \cdots Q^\rho t_0.$$

Consider the product

$$t^{\underline{\epsilon}} e^{\underline{k}} := t_0^{\epsilon_0} t_1^{\epsilon_1} \cdots e_1^{k_1} e_2^{k_2} \cdots \in \mathbf{H}_\star(\Omega^\rho S^{\rho+1})$$

with $\epsilon_i \in \{0, 1\}$ and $k_i \geqslant 0$. We compute

$$\Phi^e(t^{\underline{\epsilon}} e^{\underline{k}}) = x_1^{2k_1+\epsilon_0} x_2^{2k_2+\epsilon_1} \cdots .$$

Mapping out of the cofiber sequence (2.1) gives a fiber sequence

$$\Omega N^\times \Omega S^{\rho+1} \to \Omega^\rho S^{\rho+1} \to \Omega S^{\rho+1} \xrightarrow{\Delta} N^\times \Omega S^{\rho+1}.$$

Upon taking fixed points we get a fiber sequence

$$\Omega^2 S^3 \xrightarrow{t} (\Omega^\rho S^{\rho+1})^{C_2} \to \Omega S^2 \xrightarrow{\text{null}} \Omega S^3$$

In particular there is an equivalence

$$(\Omega^\rho S^{\rho+1})^{C_2} \simeq \Omega S^2 \times \Omega^2 S^3.$$

and we have

$$H_*(\Omega^\rho S^{\rho+1})^{C_2} \cong P[y] \otimes P[t(x_1), t(x_2), \ldots]$$

where y is the image of the fundamental class under the map

$$S^1 \to (\Omega^\rho S^{\rho+1})^{C_2}.$$

It follows that

$$\Phi^{C_2}(t^{\underline{\epsilon}} e^{\underline{k}}) = y^{\epsilon_0+2\epsilon_1+4\epsilon_2+\cdots} t(x_1)^{k_1} t(x_2)^{k_2} \cdots .$$

Thus the set

$$\{t^{\underline{\epsilon}} e^{\underline{k}}\} \subset \mathbf{H}_\star X$$

satisfies the hypotheses of Lemma 2.8, and the result follows. □

5. The equivariant Mahowald theorem

In order to prove Theorem 1.2 we will need to establish a Thom isomorphism

$$\mathbf{H}_\star(\Omega^\rho S^{\rho+1})^{\tilde{\mu}} \cong \mathbf{H}_\star \Omega^\rho S^{\rho+1}.$$

We will do so in two steps. Recall that an E_0-algebra is just a spectrum X equipped with a map $S^0 \to X$. Let $\mathrm{Free}^*_{E_\rho} : \mathrm{Alg}_{E_0}(\mathrm{Sp}^{C_2}) \to \mathrm{Alg}_{E_\rho}(\mathrm{Sp}^{C_2})$ denote a homotopical left adjoint to the forgetful functor. An explicit model for this functor is the homotopy pushout of E_ρ-algebras:

$$\begin{array}{ccc} \mathrm{Free}_{E_\rho}(S^0) & \longrightarrow & \mathrm{Free}_{E_\rho}(X) \\ \downarrow & & \downarrow \\ S^0 & \longrightarrow & \mathrm{Free}^*_{E_\rho}(X) \end{array}$$

We will need the following theorem.

Theorem 5.1. *Let $f : X \to B_{C_2}O$ classify a virtual bundle of dimension zero and denote by $\tilde{f} : \Omega^\rho\Sigma^\rho X \to B_{C_2}O$ the associated Ω^ρ-map. Then there is a canonical equivalence of E_ρ-algebras in* Sp^{C_2}

$$\mathrm{Free}^*_{E_\rho}(X^f) \cong (\Omega^\rho\Sigma^\rho X)^{\tilde{f}} .$$

Proof. Combine the equivariant approximation theorem [GM17, RS00] with Theorem IX.7.1 and Remark X.6.4 of [LMSM86]. □

Remark 5.2. The non-equivariant version of Theorem 5.1 was first observed by Mark Mahowald, and then proven by Lewis. A nice modern account in the non-equivariant setting via universal properties can be found in [AB14].

Proposition 5.3. *There is a Thom isomorphism*

$$\mathbf{H}_\star(\Omega^\rho S^{\rho+1})^{\tilde{\mu}} \cong \mathbf{H}_\star \Omega^\rho S^{\rho+1}.$$

Proof. Let $\mathrm{Free}^*_{E_\rho,\mathbf{H}} : \mathrm{Alg}_{E_0}(\mathrm{Mod}_{\mathbf{H}}) \to \mathrm{Alg}_{E_\rho}(\mathrm{Mod}_{\mathbf{H}})$ denote a homotopical left adjoint to the forgetful functor. Along with the previous theorem, we will need two facts:

(1) $\mathbf{H} \wedge (-) : \mathrm{Sp}^{C_2} \to \mathrm{Mod}_{\mathbf{H}}$ is symmetric monoidal.
(2) There is a Thom isomorphism $\mathbf{H} \wedge (S^1)^\mu \cong \mathbf{H} \wedge S^1_+$.

The proposition is now proved by the following string of equivalences:

$$\begin{aligned}
\mathbf{H} \wedge \left(\Omega^\rho \Sigma^\rho S^1\right)^{\tilde{\mu}} &\cong \mathbf{H} \wedge \mathrm{Free}^*_{E_\rho}\left((S^1)^\mu\right) && \text{by Theorem 5.1}\\
&\cong \mathrm{Free}^*_{E_\rho,\mathbf{H}}\left(\mathbf{H} \wedge (S^1)^\mu\right) && \text{by (1)}\\
&\cong \mathrm{Free}^*_{E_\rho,\mathbf{H}}\left(\mathbf{H} \wedge S^1_+\right) && \text{by (2)}\\
&\cong \mathbf{H} \wedge \mathrm{Free}^*_{E_\rho}\left(S^1_+\right) && \text{by (1)}\\
&\cong \mathbf{H} \wedge \Omega^\rho \Sigma^\rho S^1_+.
\end{aligned}$$

□

Proof of Theorem 1.2. The Thom class is represented by a map

$$(\Omega^\rho S^{\rho+1})^{\tilde{\mu}} \to \mathbf{H}.$$

We wish to show this map is an isomorphism on $\mathbf{H}_\star$. The homology of $\mathbf{H}$ is the C_2-equivariant Steenrod algebra, computed in [HK01] to be

$$\mathbf{H}_\star \mathbf{H} = \mathbf{H}_\star[\tau_0, \tau_1, \cdots, \xi_1, \xi_2, \cdots]/(\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1})$$

with

$$\begin{aligned}
|\tau_i| &= 2^i\rho - \sigma,\\
|\xi_i| &= (2^i - 1)\rho.
\end{aligned}$$

It suffices to show it is surjective, since the two homologies are abstractly isomorphic and of finite type. Observe that the composite

$$M(2) \simeq (S^1)^\mu \to (\Omega^\rho S^{\rho+1})^{\tilde{\mu}} \to \mathbf{H}$$

hits τ_0. Everything is hit then, by [Wil17, Thm. 5.4]. □

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