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Erratum

# Addendum to "A new proof of the Bott periodicity theorem" [Topology Appl. 119 (2002) 167–183] ☆

Mark Behrens [1]

*Department of Mathematics, MIT, Cambridge, MA 02139, USA*

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**Abstract**

A. Elmendorf has found an error in the approach to Lemmas 2.2 and 2.3 of "A new proof of the Bott periodicity theorem" (Topology and its Applications, 2002, 167–183). There are also errors in the definitions of the maps in Sections 4.2 and 4.5. In this paper we supply corrections to these errors. We also sketch a major simplification of the argument proving real Bott periodicity, unifying the eight quasifibrations appearing in the real case, using Clifford algebras.


*MSC:* primary 55R45; secondary 55R65

*Keywords:* Clifford modules; Bott periodicity; Homotopy groups; Spaces and groups of matrices

In Section 1 we make a correction to the definition of a mapping used in Lemmas 2.2 and 2.3 of [2]. The original error was pointed out to the author by Tony Elmendorf. We also correct some flaws in the definition of the maps $p_W$ of Sections 4.2 and 4.5 of [2].

We also take this opportunity to explain how each of the eight quasifibrations arising in the approach to real Bott periodicity given in [2] may be unified, in the context of Clifford algebras. This has the added benefit of explaining real Bott periodicity in terms of the periodicity of Clifford modules, and directly links our approach to work of Atiyah et al. [1]. Each of the quasifibrations of [2] is the instance of a general quasifibration relating certain spaces of Clifford structures. So while we are providing corrections to Sections 4.2

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*E-mail address:* mbehrens@math.mit.edu (M. Behrens).
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and 4.5, we are also inviting the reader to skip Section 4 of [2] altogether in favor of the Clifford algebra approach given in this note.

In Section 2, we introduce the spaces of Clifford extensions $X(n,\mathcal{U})$, and explain how they may be identified with the various homogeneous spaces which appear in the real Bott periodicity theorem. In Section 3, we outline a proof following the methods of [2] that there is a quasifibration

$$X(n+1,\mathcal{U}) \to E(n,\mathcal{U}) \xrightarrow{p} X(n,\mathcal{U}).$$

These spaces $E(n,\mathcal{U})$ will be contractible, thus proving Bott periodicity. Each of the separate arguments of [2] are special cases of this general argument. Section 2 is independent of [2]. Section 3 may be read as a terse proof of the real and complex Bott periodicity theorems, with the exception of occasional references to specific arguments given in [2].

## 1. Corrections to [2]

### 1.1. *The definition of $\Gamma_{W,V}$ in Section 2*

Tony Elmendorf has pointed out to the author that the definition of the map

$$\Gamma_{W,V} : \mathcal{I}(W,V) \to \mathrm{Map}\big(G(W), G(V)\big)$$

preceding Lemma 2.2 is not sound. Here $W$ and $V$ are countably infinite dimensional inner product spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The space $\mathcal{I}(W,V)$ is the space of linear isometries from $W$ to $V$. The spaces $G(W)$ and $G(V)$ are the groups of finite type isometric linear automorphisms of $W$ and $V$, respectively. These groups are isomorphic to $O$, $U$, or $Sp$, depending on the ground ring.

The problem is that if $V$ is infinite dimensional, then given an infinite subspace $V_0$ of $V$, the containment

$$V_0 \oplus V_0^{\perp} \subseteq V \tag{1.1}$$

is *not* necessarily an equality. Elmendorf points out that if one takes $V = \mathbb{R}^\infty$ with orthonormal basis $\{e_i\}$, then for the subspace $V_0$ spanned by $\{e_i + e_{i+1}\}$, the containment (1.1) is not an equality. Of course, (1.1) is an equality if $V_0$ is finite dimensional. The definition of $\Gamma_{W,V}$ given in [2] incorrectly relied on (1.1) always being an equality.

We give a correct definition of $\Gamma_{W,V}$. The finite type assumption implies that given an element $X$ of $G(W)$, there exists a finite dimensional subspace $W_0 \subseteq W$ and a transformation $X_0 \in G(W_0)$, so that

$$X = X_{W_0} \oplus I_{W_0^{\perp}}$$

under the orthogonal decomposition $W = W_0 \oplus W_0^{\perp}$. Then, given a linear isometry $\phi : W \to V$, the induced element $\phi_*(X)$ is given by

$$\phi_*(X) = \phi_{W_0} X \phi_{W_0}^{-1} \oplus I_{\phi(W_0)^{\perp}}$$

under the orthogonal decomposition $V = \phi(W_0) \oplus \phi(W_0)^{\perp}$. The definition of $\phi_*(X)$ is easily seen to be independent of the choice of $W_0$. With this definition of $\phi_*$, Lemmas 2.2 and 2.3 hold.

### 1.2. The definition of the map $p_W$ of Section 4.2

The definition of the map $p_W : E(W) \to O/U(W)$ preceding Proposition 4.5 is incorrect, as it is not compatible with the proof of Lemma 4.9. Here $E(W)$ was defined by

$$E(W) = \{A \mid A \text{ is conjugate linear and } \sigma(A) \subseteq [-i, i]\} \subseteq \mathfrak{o}(W).$$

We recall the statement of Lemma 4.9 for the reader's convenience.

**Lemma 4.9 of [2].** *Suppose that $W \subset \mathcal{U}$ is a finite dimensional quaternionic space. Let $X$ be a special representative of the class $[X] \in SO/U(W)$. Then $p_W^{-1}([X]) = U/Sp(\ker(X^2 - I))$.*

In the proof of Lemma 4.9, it is used that $p_W(A)$ is a special representative of $[X]$, but the factor of $i$ in the definition of $p_W$ makes this assertion false.

The map $p_W : E(W) \to O/U(W)$ should be defined by

$$p_W(A) = \left[ j \exp\left(\frac{\pi}{2} A\right)\right]$$

which we are regarding as an element of the *right* coset space $O/U(W)$.

The following lemma is proved by the same algebraic manipulations that prove Lemma 4.6 of [2].

**Lemma 1.1.** *Suppose that $Y$ and $Z$ in $O(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $jY = XZ$ if and only if $-Y^2 = Z^2$.*

The proof of Lemma 4.9 of [2] then proceeds as written, since our new definition of $p_W$ combined with Lemma 1.1 implies that $p_W(A) = X$ if and only if $-\exp(\pi A) = X^2$.

### 1.3. The definition of the map $p_W$ of Section 4.5

In the sentence immediately following the proof of Proposition 4.17 of [2], "$U/O(W)$" should be replaced with "$Sp/U(W)$".

The definition of the map $p_W : E(W) \to Sp/U(W)$ of Section 4.5 suffers the same deficiency as in Section 4.2, and this deficiency is fixed in exactly the same manner. Namely, the map $p_W$ is not defined correctly to make the proof of Lemma 4.20 work correctly. We recall the statement of this lemma.

**Lemma 4.20 of [2].** *Let $W \subset \mathcal{U}$ be a finite dimensional right quaternionic subspace. For a special representative $X$ of $[X] \in Sp/U(W)$, we have $p_W^{-1}([X]) \cong U/O(\ker(X^2 - I))$.*

The definition of the map $p_W$ immediately preceding Proposition 4.17 of [2] should altered to read

$$p_W(A) = \left[ j \exp\left(\frac{\pi}{2} A\right)\right]$$

which we are regarding as an element of the *right* coset space $Sp/U(W)$.

One has the following lemma, analogous to Lemma 1.1. (Recall that in Section 4.5 of [2], the group $Sp(W)$ was defined to be the collection of all *right* quaternion linear isometries of $W$, and $U(W)$ was the subgroup of right quaternion linear, left complex linear isometries.)

**Lemma 1.2.** *Suppose that $Y$ and $Z$ in $Sp(W)$ satisfy $-iYi = Y^{-1}$ and $-iZi = Z^{-1}$. Then there is an $X \in U(W)$ such that $jY = XZ$ if and only if $-Y^2 = Z^2$.*

Then, in the proof of Lemma 4.20, the new definition of $p_W$ together with Lemma 1.2, implies that $p_W(A) = [X]$ if and only if $-\exp(\pi A) = X^2$, and the rest of the proof proceeds as written.

## 2. Spaces of Clifford structures

We now explain how the ad hoc methods of Section 4 of [2] may be united in the context of Clifford algebras. Fix a real inner product space $W$. Let $C_n$ be the Clifford algebra generated by $\mathbb{R}^n$ with the standard metric. It is a real algebra on generators $e_1, \ldots, e_n$ subject to the relations

$$e_i^2 = -1,$$
$$e_i e_j = -e_j e_i, \quad i \neq j.$$

Define a *$C_n$-structure* on $W$ to be an (ungraded) $C_n$-module structure over $\mathbb{R}$ such that the generators $e_i$ act by isometries. If $W$ is given a $C_n$-structure, let $O_{C_n}(W) \subseteq O(W)$ be the collection of isometries of $W$ which preserve the $C_n$-structure.

Suppose that $W$ is given a $C_{n-1}$ structure. A $C_n$ *extension* is a $C_n$-structure which restricts to the given $C_{n-1}$-structure under the inclusion $C_{n-1} \hookrightarrow C_n$. Observe that to give a $C_n$-extension is to give an isometry $e_n$ of $W$ such that

$$e_n^2 = -I_W,$$
$$e_i e_n = -e_n e_i, \quad 0 \leqslant i < n.$$

Let $X(n, W)$ be the space of $C_n$-extensions on $W$, thought of as a subspace of $O(W)$. The group $O_{C_{n-1}}(W)$ acts on $X(n, W)$ by means of conjugation. Given $Y \in O_{C_{n-1}}(W)$, and $e_n \in X(n, W)$, the action is given by

$$Y : e_n \mapsto Y e_n Y^{-1}.$$

Clearly, the stabilizer of $e_n$ in $O_{C_{n-1}}(W)$ is $O_{C_n}(W)$, so the $e_n$ orbit is given by

$$X(n, W)_{e_n} = O_{C_{n-1}}(W)/O_{C_n}(W_{e_n})$$

where $W_{e_n}$ is given the $C_n$-structure corresponding to the $C_n$-extension $e_n$.

Given a $C_n$-structure on $W$, the module $W$ breaks up into an orthogonal direct sum of irreducible $C_n$-submodules

$$W = W_1 \oplus \cdots \oplus W_k.$$

We define $\dim_{C_n}(W)$ to be the number $k$.

If $e_n$ and $f_n$ are two $C_n$-extensions for which the $C_n$-modules $W_{e_n}$ and $W_{f_n}$ are isomorphic, then there exists an isometry $Y \in O_{C_{n-1}}(W)$ so that

$$Ye_n = f_nY.$$

It follows that $f_n$ is in the orbit of $e_n$. If $n \not\equiv 3 \pmod 4$, then $C_n$ has only one isomorphism class of irreducible modules. Thus, we have

**Lemma 2.1.** *If $n \not\equiv 3 \pmod 4$, then given any $C_n$-extension $e_n$, we have*

$$X(n, W)_{e_n} = X(n, W).$$

Suppose that we have $n \equiv 3 \pmod 4$. Then the various $e_n$-orbits correspond to the path components of $X(n, W)$. Define a volume element $\omega \in C_n$ by

$$\omega = e_1 \cdots e_n.$$

Then $\omega^2 = 1$, and $W$ breaks up as the orthogonal direct sum of its $+1$ and $-1$ eigenspaces under $\omega$-multiplication.

$$W = W^+ \oplus W^-.$$

Let $\mathcal{U}$ be a (countable infinite dimensional) real inner product space with a $C_n$-structure which contains countably many copies of each irreducible $C_n$-module as a direct summand. We shall call such a $\mathcal{U}$ a *complete $C_n$-universe*. Define spaces

$$X(n, \mathcal{U}) = \varinjlim X(n, W)$$

where the colimit is taken over finite dimensional $C_n$-submodules $W$ of $\mathcal{U}$ by extending by the given $C_n$-extension $e_n$.

We introduce one last bit of notation. Suppose that $\mathbb{K}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let $\mathbb{K}(n)$ denote the algebra of $n \times n$ matrices with entries in $\mathbb{K}$. Let $\pi_1 \in \mathbb{K}(n)$ be the projection onto the first component. Its matrix has a 1 in the $(1, 1)$-position, and zeroes elsewhere. We shall denote the image $\pi_1(W)$ by $W/n$.

Table 1 explains why the spaces $X(n, \mathcal{U})$ are important. They are the various loop spaces of $BO \times \mathbb{Z}$. Note that our use of the complete universe is necessary so that $X(3, \mathcal{U}) = BSp \times \mathbb{Z}$ and $X(7, \mathcal{U}) = BO \times \mathbb{Z}$.

We remark that this analysis carries over to the complex case to simultaneously prove complex Bott periodicity. One just replaces all real inner product spaces with complex inner product spaces, and the Clifford algebras $C_n$ with their complex analogs $C_n^{\mathbb{C}}$. The corresponding spaces $X^{\mathbb{C}}(n, \mathcal{U})$ are also given in Table 1.

Observe that there are Morita equivalence homeomorphisms

$$X(n, W) \approx X(n + 8, 16W)$$

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Table 1
The spaces $X(n,\mathcal{U})$

| $n$ | $C_n$ | $O_{C_n}(W)$ | $X(n,W)$ | $X(n,\mathcal{U})$ |
|---|---|---|---|---|
| 0 | $\mathbb{R}$ | $O(W)$ | – | – |
| 1 | $\mathbb{C}$ | $U(W)$ | $O(W)/U(W)$ | $O/U$ |
| 2 | $\mathbb{H}$ | $Sp(W)$ | $U(W)/Sp(W)$ | $U/Sp$ |
| 3 | $\mathbb{H}\oplus\mathbb{H}$ | $Sp(W^+)\times Sp(W^-)$ | $BSp(W)$ | $BSp\times\mathbb{Z}$ |
| 4 | $\mathbb{H}(2)$ | $Sp(W/2)$ | $Sp(W^-)$ | $Sp$ |
| 5 | $\mathbb{C}(4)$ | $U(W/4)$ | $Sp(W/2)/U(W/4)$ | $Sp/U$ |
| 6 | $\mathbb{R}(8)$ | $O(W/8)$ | $U(W/4)/O(W/8)$ | $U/O$ |
| 7 | $\mathbb{R}(8)\oplus\mathbb{R}(8)$ | $O(W/8^+)\times O(W/8^-)$ | $BO(W/8)$ | $BO\times\mathbb{Z}$ |
| 8 | $\mathbb{R}(16)$ | $O(W/16)$ | $O(W/8^-)$ | $O$ |
| $n$ | $C_n^{\mathbb{C}}$ | $U_{C_n^{\mathbb{C}}}(W)$ | $X^{\mathbb{C}}(n,W)$ | $X^{\mathbb{C}}(n,\mathcal{U})$ |
| 0 | $\mathbb{C}$ | $U(W)$ | – | – |
| 1 | $\mathbb{C}\oplus\mathbb{C}$ | $U(W^+)\times U(W^-)$ | $BU(W)$ | $BU\times\mathbb{Z}$ |
| 2 | $\mathbb{C}(2)$ | $U(W/2)$ | $U(W^-)$ | $U$ |

which will yield Bott periodicity. We also remark that we may extend the definition of our spaces of Clifford extensions to $X(-n,W)$ for $n\geqslant 0$. If $C_{p,q}$ is the Clifford algebra generated by $\mathbb{R}^{p+q}$ with the standard inner product of type $(p,q)$, then for a space $W$ with a $C_{0,n+1}$-structure, we define $X(-n,W)$ to be the space of $C_{1,n+1}$-extensions on $W$.

One could also work with $\mathbb{Z}/2$-graded modules instead of ungraded modules. Everything we have done would go through with a degree shift. Note that graded $C_n$-modules are the same thing as ungraded $C_{n,1}$-modules.

## 3. The general quasifibration

We will prove the following theorem, which is Bott periodicity.

**Theorem 3.1.** *Let $\mathcal{U}$ be a complete $C_{n+1}$-universe. Then there exists a quasifibration*

$$X(n+1,\mathcal{U})\to E(n,\mathcal{U})\xrightarrow{p} X(n,\mathcal{U})$$

*whose total space is contractible. Therefore there is a weak equivalence*

$$\Omega X(n,\mathcal{U})\simeq X(n+1,\mathcal{U}).$$

The quasifibration $p$ of Theorem 3.1 is the colimit of a collection of maps

$$p_W: E(n,W)\to X(n,W)$$

for each finite dimensional $C_{n+1}$-submodule $W$ of $\mathcal{U}$. Define $E(n,W)$ as space of skew-symmetric transformations

$$E(n,W)=\{A\in\mathfrak{o}(W)\colon\ \sigma(A)\subseteq[-i,i],\ e_nA=-Ae_n,\ e_iA=Ae_i,\ 1\leqslant i<n\}.$$

Here $\sigma(A)$ is the spectrum of $A$, thinking of it as an element of $\mathfrak{u}(W\otimes_{\mathbb{R}}\mathbb{C})$. Note that the commutation relations we have imposed on elements of $E(n,W)$ force them to lie in the orthogonal complement of the Lie algebra $\mathfrak{o}_{C_n}(W)$ in $\mathfrak{o}_{C_{n-1}}(W)$.

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Define the map $p_W : E(n, W) \to X(n, W)$ by

$$p_W : A \mapsto -\exp\left(\frac{\pi}{2}A\right)e_n \exp\left(\frac{\pi}{2}A\right)^{-1}.$$

Observe that $e_{n+1}e_n$ may be regarded as an element of $E(n, W)$, and that we have

$$\begin{aligned} p_W(e_{n+1}e_n) &= -\exp\left(\frac{\pi}{2}e_{n+1}e_n\right)e_n \exp\left(\frac{\pi}{2}e_{n+1}e_n\right) \\ &= -\exp(\pi e_{n+1}e_n)e_n \\ &= e_n. \end{aligned}$$

The last equality follows from the fact that $(e_{n+1}e_n)^2 = -I$, so the eigenvalues of $e_{n+1}e_n$ are contained in $\{\pm i\}$.

For any $C_{n+1}$-space $V$ contained in $W^\perp$, define inclusions $\iota_{W,V} : X(n, W) \hookrightarrow X(n, W \oplus V)$ and $\tilde{\iota}_{W,V} : E(n, W) \hookrightarrow E(n, W \oplus V)$ which for $f_n \in X(n, W)$ and $A \in E(n, W)$, are given by

$$\begin{aligned} &\iota_{W,V} : f_n \mapsto f_n \oplus e_n|_V, \\ &\tilde{\iota}_{W,V} : A \mapsto A \oplus e_{n+1}e_n|_V. \end{aligned}$$

These inclusions are compatible with $p_W$, so that we may define

$$p : E(n, \mathcal{U}) \to X(n, \mathcal{U})$$

to be the colimit of the maps $p_W$.

We now endeavor to identify the fiber of $p_W$. Note that for $A \in E(n, W)$, the matrix $Y = \exp(\frac{\pi}{2}A)$ has the properties:

$$\begin{aligned} &e_i Y = Y e_i, \quad 1 \leqslant i < n, \\ &e_n Y = Y^{-1}e_n. \end{aligned}$$

The first property implies that $p_W$ takes values in $X(n, W)$. The second property allows us to apply the following trivial lemmas.

**Lemma 3.2.** *Suppose that $Y$ and $Z$ in $O(W)$ satisfy $e_nY = Y^{-1}e_n$ and $e_nZ = Z^{-1}e_n$. Then we have $-Ye_nY^{-1} = -Ze_nZ^{-1}$ if and only if $Y^2 = Z^2$.*

**Lemma 3.3.** *Given $f_n$ in $X(n, W)$, we have $p_W(A) = f_n$ if and only if $A$ satisfies* $\exp(\pi A) = f_ne_n$.

**Proof.** Given an element $f_n$ of $X(n, W)$, such that $f_n = -Ye_nY^{-1}$ we may recover $Y^2 = f_ne_n$. Thus the lemma follows from Lemma 3.2. □

**Lemma 3.4.** *For $f_n$ an element of $X(n, W)_{e_n}$, the fiber of $p_W$ over $f_n$ is given by*

$$p_W^{-1}(f_n) = X\big(n+1, \ker(e_n - f_n)\big).$$

*Here* $\ker(e_n - f_n) \subset W$ *is a $C_n$-submodule with respect to the given $C_n$-structure on $W$.*

**Proof.** Regarding the matrix $f_n e_n$ as an element of $U(W \otimes_{\mathbb{R}} \mathbb{C})$, it has a spectral decomposition into a sum of projections

$$f_n e_n = -\pi_V + \sum_l \lambda_l \pi_{V_l}$$

where $V = \ker(f_n e_n + I) = \ker(e_n - f_n)$ and $\lambda_l \neq -1$. Let $A$ be an element of $p_W^{-1}(f_n)$. By Lemma 3.3, we have $f_n e_n = \exp(\pi A)$. Regarding $A$ as an element of $\mathfrak{u}(W \otimes_{\mathbb{R}} \mathbb{C})$, it has a spectral decomposition

$$A = i\pi_{V'} - i\pi_{V''} + \sum_l \mu_l \pi_{V_l}$$

where $\mu_l$ are the unique elements of $(-i, i)$ for which $e^{\pi\mu_l} = \lambda_l$ and $V' \oplus V'' = V$. It follows that when restricted to $V$, $A^2 = -I$. One easily checks that given this and the commutation relations associated to being an element of $E(n, W)$, the transformation $f_{n+1} = e_n A$ is a $C_{n+1}$-extension on $V = \ker(e_n - f_n)$.

Conversely, given $f_{n+1} \in X(n+1, V)$, then $(f_{n+1}e_n)^2 = -I_V$, so on $V$ the transformation $f_{n+1}e_n$ has a spectral decomposition of the form $i\pi_{V'} - i\pi_{V''}$ where $V = V' \oplus V''$. We then define the corresponding $A \in p_W^{-1}(f_n)$ by

$$A = i\pi_{V'} - i\pi_{V''} + \sum_l \mu_l \pi_{V_l}$$

where the $\mu_l$ are given as before. □

Observe that elements of $X(n, \mathcal{U})$ may be regarded as $C_n$-extensions $f_n$ on $\mathcal{U}$ for which there exists a finite dimensional subspace $W(f_n, e_n)$ such that

$$W(f_n, e_n)^{\perp} = \ker(e_n - f_n).$$

We shall say that such a $C_n$-structure $f_n$ is *virtually equivalent* to $e_n$. Note that virtual equivalence is an equivalence relation. We have shown that the map $p : E(n, \mathcal{U}) \to X(n, \mathcal{U})$ has fibers

$$p^{-1}(f_n) = X\big(n+1, \ker(e_n - f_n)\big) = X\big(n+1, W(e_n, f_n)^{\perp}\big)$$

for $f_n$ virtually equivalent to $e_n$.

**Remark.** The map $p$ surjects onto the path component of $e_n$, using the fact that path components of $O_{C_{n-1}}(W)/O_{C_n}(W)$ are geodesically complete. If $f_n \in X(n, \mathcal{U})$ is in the image of $p$, then $\ker(e_n - f_n)$ will admit a $C_{n+1}$-extension which is the restriction of a $C_{n+1}$ extension on $\mathcal{U}$ which is virtually equivalent to $e_{n+1}$. In fact, if $f_n = -Ye_nY^{-1}$, for $Y$ having the property that $e_iY = Ye_i$ for $1 \leqslant i < n$ and $e_nY = Y^{-1}e_n$, then $f_{n+1} = -Ye_{n+1}Y^{-1}$ is such a $C_{n+1}$-extension on $\mathcal{U}$, for which $\ker(e_n - f_n)$ and $W(e_n, f_n)$ are $C_{n+1}$-submodules. The space $X(n+1, \ker(e_n - f_n))$ is the space of $C_{n+1}$-extensions on $\ker(e_n - f_n)$ which are virtually equivalent to $f_{n+1}$.

We will apply the Dold–Thom theorem to prove that $p$ is a quasifibration, thus completing the proof of Theorem 3.1. Define a filtration on $X(n, \mathcal{U})_{e_n}$ by setting

$$F_k X(n, \mathcal{U})_{e_n} = \big\{ f_n \colon \dim_{C_{n+1}} W(f_n, e_n) \leqslant k \big\}.$$

The proof that the filtration annuli $F_k X(n,\mathcal{U}) - F_{k-1}X(n,\mathcal{U})$ are distinguished follows the same line of argument as Lemma 3.3 of [2]. The essential point is that for finite dimensional $C_{n+1}$-spaces $W$ with a $C_{n+1}$-subspace $V$, the projection

$$O_{C_n}(W)/O_{C_{n+1}}(V) \times O_{C_n}\left(V^{\perp}\right) \to O_{C_n}(W)/O_{C_n}(V) \times O_{C_n}\left(V^{\perp}\right)$$

is a fibration.

We may define neighborhoods $N_k$ of $F_{k-1}X(n,\mathcal{U})$ in $F_k X(n,\mathcal{U})$ by

$$N_k = \{f_n\colon \dim_{C_{n+1}} \mathrm{Eig}_{\exp(i\pi[-1/2,1/2])} f_n e_n < k\}$$

where the eigenspace is given the $C_{n+1}$-extension $f_{n+1}$ as in the preceding remark.

Letting $f:[-i,i] \to [-i,i]$ be the function given by

$$f(x) = \begin{cases} -i, & \mathrm{Im}(x) < -1/2, \\ 2x, & -1/2 \leqslant \mathrm{Im}(x) \leqslant 1/2, \\ i, & \mathrm{Im}(x) > 1/2. \end{cases}$$

Then $f$ is homotopic to Id rel $\{-i,i\}$. Let $H$ be such a homotopy and define $h: S^1 \times I \to S^1$ so that the following diagram commutes.

$$\begin{array}{ccc} [-i,i] & \xrightarrow{H_t} & [-i,i] \\ {\scriptstyle e^{\pi(\cdot)}}\downarrow & & \downarrow{\scriptstyle e^{\pi(\cdot)}} \\ S^1 & \xrightarrow[h_t]{} & S^1 \end{array}$$

Then the functional calculus (see the discussion preceding Lemma 3.4 of [2]) gives a homotopy $H_t: E(n,\mathcal{U}) \to E(n,\mathcal{U})$ which covers $h_t: X(n,\mathcal{U}) \to X(n,\mathcal{U})$ by

$$H_t: A \mapsto H_t(A),$$

$$h_t: f_n \mapsto -h_t(f_n e_n)e_n.$$

The hypotheses of the Dold–Thom theorem require that the induced map $H_0: p^{-1}(f_n) \to p^{-1}(h_0(f_n))$ induces a homotopy equivalence on fibers. This follows from the following lemma.

**Lemma 3.5.** *Suppose that $W$ and $V$ are orthogonal finite-dimensional $C_{n+1}$-subspaces of $\mathcal{U}$. Then the map*

$$X\left(n+1, (V \oplus W)^{\perp}\right) \to X\left(n+1, W^{\perp}\right)$$

*given by $f_{n+1} \mapsto f_{n+1} \oplus e_{n+1}|_V$ is a weak equivalence.*

**Proof.** Since the spaces $X(n+1,\mathcal{V})$ are given as homogeneous spaces involving the groups $O$, $U$, or $Sp$ (see Table 1), this theorem follows directly from Lemma 2.3 of [2]. □

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