1

Topological modular and automorphic forms

Mark Behrens

CONTENTS

1.1 Introduction

The spectrum of topological modular forms (TMF) was first introduced by Hopkins and Miller [HM14b], [Hop95], [Hop02], and Goerss and Hopkins constructed it as an E_∞ ring spectra (see [Beh14]). Lurie subsequently gave a conceptual approach to TMF using his theory of spectral algebraic geometry [Lur09]. Lurie's construction relies on a general theorem [Lur18a], [Lur18b], which was used by the author and Lawson to construct spectra of topological automorphic forms (TAF) [BL10].

The goal of this article is to give an accessible introduction to TMF and TAF spectra. Besides the articles mentioned above, there already exist many excellent such surveys (see [HM14a], [Rez07], [Law09], [Goe09], [Goe10], [DFHH14]). Our intention is to give an account which is somewhat complementary to these existing surveys. We assume the reader knows about the stable homotopy category, and knows some basic algebraic geometry, and attempt to give the reader a concrete understanding of the big picture while deemphasizing many of the technical details. Hopefully, the reader does not find the inevitable sins of omission to be too grievous.

In Section 1.2 we recall the definition of a complex orientable ring spectrum E, and its associated formal group law F_E. We then explain how an algebraic group G also gives rise to a formal group law $\widehat{G}$, and define elliptic cohomology theories to be complex orientable cohomology theories whose formal group laws arise from elliptic curves. We explain how a theorem of Landweber proves the existence of certain *Landweber exact* elliptic cohomology theories.

We proceed to define topological modular forms in Section 1.3. We first begin by recalling the definition of classical modular forms as sections of powers of a certain line bundle on the compactification $\overline{\mathcal{M}}_{ell}$ of the moduli stack $\mathcal{M}_{ell}$ of elliptic curves. Then we state a theorem of Goerss-Hopkins-Miller, which states that there exists a sheaf of E_∞ ring spectra $\mathcal{O}^{top}$ on the étale site of $\overline{\mathcal{M}}_{ell}$ whose sections over an affine

$$\mathrm{spec}(R) \xrightarrow[\text{etale}]{C} \mathcal{M}_{ell}$$

recover the Landweber exact elliptic cohomology theory associated to the elliptic curve C it classifies. The spectrum Tmf is defined to be the global sections of this sheaf:

$$\mathrm{Tmf} := \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}).$$

We compute $\pi_*\mathrm{Tmf}[1/6]$, and use that to motivate the definition of connective topological modular forms (tmf) as the connective cover of Tmf, and periodic topological modular forms (TMF) as the sections over the non-compactified moduli stack:

$$\mathrm{TMF} := \mathcal{O}^{top}(\mathcal{M}_{ell}).$$

The homotopy groups of TMF at the primes 2 and 3 are more elaborate.

While we do not recount the details of these computations, we do indicate the setup in Section 1.4, and state the results in a form that we hope is compact and understandable. The computation of $\pi_* \mathrm{Tmf}_{(p)}$ for $p = 2, 3$ is then discussed in Section 1.5. We introduce the notion of the height of a formal group law, and use it to create a computable cover of the compactified moduli stack $(\overline{\mathcal{M}}_{ell})_{\mathbb{Z}_{(p)}}$. By taking connective covers, we recover the homotopy of groups of $\mathrm{tmf}_{(p)}$.

In Section 1.6, we go big picture. We explain how complex cobordism associates to certain ring spectra E a stack

$$\mathcal{X}_E \to \mathcal{M}_{fg}$$

over the moduli stack of formal group laws. The sheaf $\mathcal{O}^{top}$ serves as a partial inverse to $\mathcal{X}_{(-)}$, in the sense that where it is defined, we have

$$\mathcal{O}^{top}(\mathcal{X}_E) \simeq E,$$
$$\mathcal{X}_{\mathcal{O}^{top}(\mathcal{U})} \simeq \mathcal{U}.$$

We describe the height filtration of the moduli stack of formal groups $\mathcal{M}_{fg}$, and explain how chromatic localizations of the sphere realize this filtration in topology. The stacks associated to chromatic localizations of a ring spectrum E are computed by pulling back the height filtration to $\mathcal{X}_E$. We then apply this machinery to Tmf to compute its chromatic localizations, and explain how chromatic fracture is closely connected to the approach to $\pi_* \mathrm{Tmf}$ discussed in Section 1.5.

We then move on to discuss Lurie's theorem, which expands $\mathcal{O}^{top}$ to the étale site of the moduli space of *p-divisible groups*. After recalling the definition, we state Lurie's theorem, and explain how his theorem simultaneously recovers the Goerss-Hopkins-Miller theorem on Morava E-theory, and the Goerss-Hopkins-Miller sheaf $\mathcal{O}^{top}$ on $\mathcal{M}_{ell}$. We then discuss a class of moduli stacks of Abelian varieties (*PEL Shimura stacks of type* $U(1, n-1)$) which give rise to spectra of topological automorphic forms [BL10].

There are many topics which should have appeared in this survey, but regrettably do not, such as the Witten orientation, the connection to 2-dimensional field theories, spectral algebraic geometry, and equivariant elliptic cohomology, to name a few. We compensate for this deficiency in Section 1.8 with a list of such topics, and references to the literature for further reading.

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1.2 Elliptic cohomology theories

Complex orientable ring spectra

Let E be a (homotopy associative, homotopy commutative) ring spectrum.

Definition 1.2.1. A *complex orientation* of E is an element

$$x \in \widetilde{E}^*(\mathbb{C}P^\infty)$$

such that the restriction

$$x|_{\mathbb{C}P^1} \in \widetilde{E}^*(\mathbb{C}P^1)$$

is a generator (as an E_*-module). A ring spectrum which admits a complex orientation is called *complex orientable*.

For complex oriented ring spectra E, the Atiyah-Hirzebruch spectral sequence collapses to give

$$E^*(\mathbb{C}P^\infty) = E_*[[x]], \tag{1.2.2}$$

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E_*[[x_1, x_2]] \tag{1.2.3}$$

where x_i is the pullback of x under the ith projection.

Consider the map

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

which classifies the universal tensor product of line bundles. Novikov [Nov67, p.853] and Quillen [Qui69] (see also [Ada74]) observed that because μ gives $\mathbb{C}P^\infty$ the structure of a homotopy commutative, homotopy associative H-space, the power series

$$F_E(x_1, x_2) := \mu^* x \in E_*[[x_1, x_2]]$$

is a (commutative, 1 dimensional) *formal group law* over E_*, in the sense that it satisfies

1. $F_E(x, 0) = F_E(0, x) = x$,
2. $F_E(x_1, F_E(x_2, x_3)) = F_E(F_E(x_1, x_2), x_3)$,
3. $F_E(x_1, x_2) = F_E(x_2, x_1)$.

Example 1.2.4. Let $E = H\mathbb{Z}$, the integral Eilenberg-MacLane spectrum. Then the complex orientation x is a generator of $H^2(\mathbb{C}P^\infty)$, and

$$F_{H\mathbb{Z}}(x_1, x_2) = x_1 + x_2.$$

This is the *additive formal group law* F_{add}.

Example 1.2.5. Let $E = KU$, the complex K-theory spectrum. Then the class

$$x := [L_{\mathrm{can}}] - 1 \in \widetilde{KU}^0(\mathbb{C}P^\infty)$$

(where L_{can} is the canonical line bundle) gives a complex orientation for KU, and

$$F_{KU}(x_1, x_2) = x_1 + x_2 + x_1 x_2.$$

This is the *multiplicative formal group law* F_{mult}.

Example (1.2.5) above is an example of the following.

Definition 1.2.6. An *even periodic* ring spectrum is a ring spectrum E so that

$$\pi_{\mathrm{odd}} E = 0 \tag{1.2.7}$$

and such that E_2 contains a unit.

It is easy to see using a collapsing Atiyah-Hirzebruch spectral sequence argument that (1.2.7) is enough to guarantee the complex orientability of an even periodic ring spectrum E. The existence of the unit in E_2 implies one can take the complex orientation to be a class

$$x \in \widetilde{E}^0(\mathbb{C}P^\infty)$$

giving

$$E^0(\mathbb{C}P^\infty) = E_0[[x]]. \tag{1.2.8}$$

It follows that in the even periodic case, for such choices of complex orientation, we can regard the formal group law F_E as a formal group law over E_0, and (1.2.8) can be regarded as saying

$$E^0(\mathbb{C}P^\infty) = \mathcal{O}_{F_E}.$$

where the latter is the ring of functions on the formal group law. Then it follows that we have a canonical identification

$$E_2 = \widetilde{E}^0(\mathbb{C}P^1) = (x)/(x)^2 = T_0^* F_E$$

Here (x) is the ideal generated by x in $E^0(\mathbb{C}P^\infty)$ and $T_0^* F_E$ is the cotangent space of F_E at 0. The even periodicity of E then gives

$$E_{2i} \xleftarrow{\cong} E_2^{\otimes_{E_0} i} = (T_0^* F_E)^{\otimes i}. \tag{1.2.9}$$

Here (1.2.9) even makes sense for i negative: since E_2 is a free E_0-module of rank 1, it is invertible (in an admittedly trivial manner), and $T_0^* F_E$ is invertible since it is a line bundle over $\mathrm{spec}(E_0)$.

Formal groups associated to algebraic groups

Formal group laws also arise in the context of algebraic geometry. Let G be a 1-dimensional commutative algebraic group over a commutative ring R. If the line bundle T_eG (over $\mathrm{spec}(R)$) is trivial, there exists a coordinate x of G at the identity $e \in G$. We shall call such group schemes *trivializable*. In this case the group structure

$$G \times G \to G$$

can be expressed locally in terms of the coordinate x as a power series

$$\widehat{G}(x_1, x_2) \in R[[x_1, x_2]].$$

The unitality, associativity, and commutativity of the group structure on G makes $\widehat{G}$ a formal group law over R. The formal groups in Examples 1.2.4 and 1.2.5 arise in this manner from the additive and multiplicative groups $\mathbb{G}_a$ and $\mathbb{G}_m$ (defined over $\mathbb{Z}$) by making appropriate choices of coordinates:

$$\widehat{\mathbb{G}}_a = F_{add},$$
$$\widehat{\mathbb{G}}_m = F_{mult}.$$

It turns out that if we choose different coordinates/complex orientations, we will still get isomorphic formal group laws. A *homomorphism* $f : F \to F'$ of formal group laws over R is a formal power series

$$f(x) \in R[[x]]$$

satisfying

$$f(F(x_1, x_2)) = F'(f(x_1), f(x_2)).$$

If the power series $f(x)$ is invertible (with respect to composition) then we say that it is an isomorphism. Clearly, choosing a different coordinate on a trivializable commutative 1-dimensional algebraic group gives an isomorphic formal group law. One similarly has the following proposition.

Proposition 1.2.10. *Suppose that x and x' are two complex orientations of a complex orientable ring spectrum E, with corresponding formal group laws F and F'. Then there is a canonical isomorphism between F and F'.*

Proof. Using (1.2.2), we deduce that $x' = f(x) \in E_*[[x]]$. It is a simple matter to use the resulting change of coordinates to verify that f is an isomorphism from F to F'. □

Remark 1.2.11. A *formal group* is a formal group law where we forget the data of the coordinate. Thus Proposition 1.2.10 is asserting that a complex orientable ring spectrum gives rise to a canonical formal group.

The only 1-dimensional connected algebraic groups over an algebraically closed field are $\mathbb{G}_a$, $\mathbb{G}_m$, and elliptic curves.[1] As we have shown that there are complex orientable ring spectra which yield the formal groups of the first two, it is reasonable to consider the case of elliptic curves.

The elliptic case

Definition 1.2.12 ([AHS01])**.** An *elliptic cohomology theory* consists of a triple

$$(E, C, \alpha)$$

where

$$\begin{aligned} E &= \text{an even periodic ring spectrum,} \\ C &= \text{a trivializable elliptic curve over } E_0, \\ (\alpha : \widehat{C} \xrightarrow{\cong} F_E) &= \text{an isomorphism of formal group laws.} \end{aligned}$$

Remark 1.2.13. Note that every elliptic curve C that admits a Weierstrass presentation

$$C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

is trivializable, since $z = x/y$ is a coordinate at $e \in C$.

For an elliptic cohomology theory (E, C, α), the map α_* gives an isomorphism

$$T_e^*C \cong T_0^*\widehat{C} \xleftarrow[\cong]{\alpha^*} T_0^*F_E.$$

It follows that we have a canonical isomorphism

$$E_{2i} \cong (T_e^*C)^{\otimes i}. \tag{1.2.14}$$

It is reasonable to ask when elliptic cohomology theories exist. This was first studied by Landweber, Ravenel, and Stong [LRS95] using the Landweber Exact Functor Theorem [Lan76]. Here we state a reformulation of this theorem which appears in [Nau07] (this perspective originates with Franke [Fra96] and Hopkins [Hop99]).

Theorem 1.2.15 (Landweber Exact Functor Theorem)**.** *Suppose that F is a formal group law over R whose classifying map*

$$\mathrm{spec}(R) \xrightarrow{F} \mathcal{M}_{fg}$$

to the moduli stack of formal groups is flat. Then there exists a unique (in the homotopy category of ring spectra) even periodic ring spectrum E with $E_0 = R$ and $F_E \cong F$.

[1] An elliptic curve over an algebraically closed field k is a smooth proper connected curve over k of genus 1, with a chosen k-point (which serves as the identity of the resulting algebraic group).

Corollary 1.2.16. *Suppose that C is a trivializable elliptic curve over R whose associated formal group law $\widehat{C}$ satisfies the hypotheses of the Landweber exact functor theorem. Then there exists an elliptic cohomology theory E_C associated to the elliptic curve C.*

Remark 1.2.17. For us, a *stack* is a functor

$$\mathrm{Rings}^{op} \to \mathrm{Groupoids}$$

which satisfies a descent condition with respect to a given Grothendieck topology. The moduli stack of formal groups $\mathcal{M}_{fg}$ associates to a ring R the groupoid whose objects are formal group schemes over R that are Zariski locally (in $\mathrm{spec}(R)$) isomorphic to the formal affine line $\widehat{\mathbb{A}}^1$, and whose morphisms are the isomorphisms of such.

Remark 1.2.18. Landweber's original formulation of his exact functor theorem did not use the language of stacks, but rather gave a simple to check explicit criterion which is equivalent to the flatness condition.

The problem with Landweber's theorem is that while it gives a functor

$$\{\text{Landweber flat formal groups}\} \to \mathrm{Ho}(\mathrm{Spectra}),$$

this functor does not refine to a point-set level functor to spectra.

1.3 Topological modular forms

Classical modular forms

Let $\mathcal{M}_{ell}$ denote the moduli stack of elliptic curves (over $\mathrm{spec}(\mathbb{Z})$). It is the stack whose groupoid of R-points is the groupoid of elliptic curves over R and isomorphisms. Consider the line bundle ω on $\mathcal{M}_{ell}$ whose fiber over an elliptic curve C is given by the cotangent space at the identity

$$\omega_C = T_e^* C.$$

The moduli stack of elliptic curves $\mathcal{M}_{ell}$ admits a compactification $\overline{\mathcal{M}}_{ell}$ [DR73] where we allow our elliptic curves to degenerate to singular curves in the form of *Néron n-gons*. The line bundle ω extends over this compactification. The space of (integral) modular forms of weight k is defined to be the global sections (see [Kat73])

$$MF_k := H^0(\overline{\mathcal{M}}_{ell}, \omega^{\otimes k}). \tag{1.3.1}$$

The complex points $\mathcal{M}_{ell}(\mathbb{C})$ admit a classical description (see, for example, [Sil94]). Let $\mathcal{H} \subseteq \mathbb{C}$ denote the upper half plane. Then we can associate to a point $\tau \in \mathcal{H}$ an elliptic curve C_τ over $\mathbb{C}$ by defining

$$C_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

Every elliptic curve over $\mathbb{C}$ arises this way. Let $SL_2(\mathbb{Z})$ act on $\mathcal{H}$ through Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Two such elliptic curves C_τ and $C_{\tau'}$ are isomorphic if and only if $\tau' = A \cdot \tau$ for some A in $SL_2(\mathbb{Z})$. It follows that

$$\mathcal{M}_{ell}(\mathbb{C}) = \mathcal{H}//SL_2(\mathbb{Z}).$$

In this language, a modular form $f \in MF_k$ can be regarded as a meromorphic function on $\mathcal{H}$ which satisfies

$$f(\tau) = (c\tau + d)^{-k} f(A \cdot \tau)$$

for every

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The condition of extending over the compactification $\overline{\mathcal{M}}_{ell}$ can be expressed over $\mathbb{C}$ by requiring that the Fourier expansion (a.k.a. *q-expansion*)

$$f(\tau) = \sum_{i \in \mathbb{Z}} a_i q^i \quad (q := e^{2\pi i \tau})$$

satisfies $a_i = 0$ for $i < 0$ (a.k.a. "holomorphicity at the cusp").

The Goerss-Hopkins-Miller sheaf

The following major result of Goerss-Hopkins-Miller [HM14b], [Beh14] gives a topological lift of the sheaf $\bigoplus_i \omega^{\otimes i}$.

Theorem 1.3.2 (Goerss-Hopkins-Miller)**.** *There is a homotopy sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on the étale site of $\overline{\mathcal{M}}_{ell}$ with the property that the spectrum of sections*

$$E_C := \mathcal{O}^{top}(\mathrm{spec}(R) \xrightarrow{C} \mathcal{M}_{ell})$$

associated to an étale map $\mathrm{spec}(R) \to \mathcal{M}_{ell}$ *classifying a trivializable elliptic curve C/R is an elliptic cohomology theory for the elliptic curve C.*

Remark 1.3.3. Since the map

$$\begin{gathered} \mathcal{M}_{ell} \to \mathcal{M}_{fg} \\ C \mapsto \widehat{C} \end{gathered}$$

is flat, it follows that every elliptic cohomology theory E_C coming from the theorem above could also have been constructed using Corollary 1.2.16. The novelty in Theorem 1.3.2 is:

1. the functor $\mathcal{O}^{top}$ lands in the point-set category of spectra, rather than in the homotopy category of spectra,
2. the spectra E_C are E_∞, not just homotopy ring spectra, and
3. the functor $\mathcal{O}^{top}$ can be evaluated on non-affine étale maps of stacks
$$\mathcal{X} \to \overline{\mathcal{M}}_{ell}.$$

Elaborating on point (3) above, the "homotopy sheaf" property of $\mathcal{O}^{top}$ implies that for any étale cover
$$\mathcal{U} = \{U_i\} \to \mathcal{X}$$
the map
$$\mathcal{X} \to \mathrm{Tot}\left(\prod_{i_0} \mathcal{O}^{top}(U_{i_0}) \Rightarrow \prod_{i_0,i_1} \mathcal{O}^{top}(U_{i_0} \times_{\mathcal{X}} U_{i_1}) \Rrightarrow \cdots \right)$$
is a equivalence. The Bousfield-Kan spectral sequence [BK73] of the totalization takes the form
$$E_1^{s,t} = \prod_{i_0,\ldots,i_s} \pi_t \mathcal{O}^{top}(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_s}) \Rightarrow \pi_{t-s}\mathcal{O}^{top}(\mathcal{X}). \tag{1.3.4}$$
Because $\overline{\mathcal{M}}_{ell}$ is a separated Deligne-Mumford stack, there exists a cover of $\mathcal{X}$ by affines, and all of their iterated pullbacks are also affine. Since every elliptic curve is locally trivializable over its base, we can refine any such cover to be a cover which classifies trivializable elliptic curves. In this context we find (using (1.2.14)) that the E_1-term above can be identified with the Čech complex
$$E_1^{s,2k} = \check{C}^s_{\mathcal{U}}(\mathcal{X}, \omega^{\otimes k})$$
and we obtain the *descent spectral sequence*
$$E_2^{s,2k} = H^s(\mathcal{X}, \omega^{\otimes k}) \Rightarrow \pi_{2k-s}\mathcal{O}^{top}(\mathcal{X}). \tag{1.3.5}$$

Non-connective topological modular forms

Motivated by (1.3.1) and (1.3.5), we make the following definition.

Definition 1.3.6. The spectrum of *(non-connective) topological modular forms* is defined to be the spectrum of global sections
$$\mathrm{Tmf} := \mathcal{O}^{op}(\overline{\mathcal{M}}_{ell}).$$

To get a feel for Tmf, we investigate the descent spectral sequence for Tmf[1/6].

Proposition 1.3.7. *We have*[2]

$$H^*((\overline{\mathcal{M}}_{ell})_{\mathbb{Z}[1/6]}, \omega^{\otimes *}) = \mathbb{Z}[1/6][c_4, c_6] \oplus \frac{\mathbb{Z}[1/6][c_4, c_6]}{(c_4^\infty, c_6^\infty)}\{\theta\}$$

where

$$\begin{aligned} c_k &\in H^0((\overline{\mathcal{M}}_{ell})_{\mathbb{Z}[1/6]}, \omega^{\otimes k}), \\ \theta &\in H^1((\overline{\mathcal{M}}_{ell})_{\mathbb{Z}[1/6]}, \omega^{\otimes -10}). \end{aligned}$$

Thus there are no possible differentials or extensions in the descent spectral sequence, and we have

$$\pi_* \mathrm{Tmf}[1/6] \cong \mathbb{Z}[1/6][c_4, c_6] \oplus \frac{\mathbb{Z}[1/6][c_4, c_6]}{(c_4^\infty, c_6^\infty)}\{\theta\}$$

with

$$\begin{aligned} |c_k| &= 2k, \\ |\theta| &= -21. \end{aligned}$$

Proof. Every trivializable elliptic curve C over a $\mathbb{Z}[1/6]$-algebra R can be embedded in $\mathbb{P}^2$, where it takes the Weierstrass form (see, for example, [Sil09, III.1])

$$C_{c_4,c_6} : y^2 = x^3 - 27c_4 x - 54c_6, \quad c_4, c_6 \in R \tag{1.3.8}$$

where the *discriminant*

$$\Delta := \frac{c_4^3 - c_6^2}{1728}$$

is invertible. The isomorphisms of elliptic curves of this form are all of the form

$$\begin{aligned} f_\lambda : C_{c_4,c_6} &\to C_{c_4',c_6'} \\ (x, y) &\mapsto (\lambda^2 x, \lambda^3 y) \end{aligned}$$

with

$$c_k' = \lambda^k c_k. \tag{1.3.9}$$

We deduce that

$$(\mathcal{M}_{ell})_{\mathbb{Z}[1/6]} = \mathrm{spec}(\mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}])//\mathbb{G}_m$$

where the $\mathbb{G}_m$-action is given by (1.3.9). The $\mathbb{G}_m$-action encodes a grading on $\mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}]$ where

$$\deg c_k := k.$$

[2]Here, we use the notation $\frac{\mathbb{Z}[1/6][c_4,c_6]}{(c_4^\infty, c_6^\infty)}\{\theta\}$ to mean that θ is highest degree non-zero class in this divisible pattern.

Using the coordinate $z = x/y$ at ∞ for the Weierstrass curve C_{c_4,c_6} (the identity for the group structure), we compute

$$f_\lambda^* dz = \lambda^{-1} dz.$$

It follows that the cohomology of $\omega^{\otimes k}$ is the kth graded summand of the cohomology of the structure sheaf of $\mathrm{spec}(\mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}])$:

$$H^s((\mathcal{M}_{ell})_{\mathbb{Z}[1/6]}, \omega^{\otimes k}) \cong H^{s,k}(\mathrm{spec}(\mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}])) \tag{1.3.10}$$

$$= \begin{cases} \mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}]_k, & s = 0, \\ 0, & s > 0. \end{cases} \tag{1.3.11}$$

We extend the above analysis to the compactification $\overline{\mathcal{M}}_{ell}$ by allowing for nodal singularities.[3] A curve C_{c_4,c_6} has a nodal singularity if and only if $\Delta = 0$ and c_4 is invertible. We therefore compute

$$\begin{aligned} &H^s((\overline{\mathcal{M}}_{ell})_{\mathbb{Z}[1/6]}, \omega^{\otimes k}) \\ &\qquad \cong H^{s,k}(\mathrm{spec}(\mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}]) \cup \mathrm{spec}(\mathbb{Z}[1/6][c_4^{\pm}, c_6])) \end{aligned}$$

as the kernel and cokernel of the map

$$\begin{matrix} \mathbb{Z}[1/6][c_4, c_6, \Delta^{-1}] \\ \oplus \\ \mathbb{Z}[1/6][c_4^{\pm}, c_6] \end{matrix} \to \mathbb{Z}[1/6][c_4^{\pm}, c_6, \Delta^{-1}].$$

$\square$

The computation above implies that the unlocalized cohomology

$$H^s(\overline{\mathcal{M}}_{ell}, \omega^{\otimes k})$$

consists entirely of 2- and 3-torsion for $s > 1$. In fact, it turns out that these groups are non-trivial for arbitrarily large values of s, resulting in 2- and 3-torsion persisting to $\pi_*\mathrm{Tmf}$. This will be discussed in more detail in Section 1.4.

Variants of Tmf

We highlight two variants of the spectrum Tmf: the *connective* and the *periodic* versions. One feature of $\pi_*\mathrm{Tmf}[1/6]$ which is apparent in Proposition 1.3.7 is that

$$\pi_k \mathrm{Tmf}[1/6] = 0, \quad -20 \le k \le -1.$$

It turns out that this gap in homotopy groups occurs in the unlocalized Tmf spectrum (see Section 1.4), and the negative homotopy groups of Tmf are

[3] A curve of the form C_{c_4,c_6} can only have nodal or cuspidal singularities [Sil09, III.1], and the nodal case is a Néron 1-gon.

related to the positive homotopy groups of Tmf by Anderson duality (at least with 2 inverted, see [Sto12]).

We therefore isolate the positive homotopy groups by defining the *connective* tmf*-spectrum* to be the connective cover

$$\mathrm{tmf} := \tau_{\geq 0}\mathrm{Tmf}.$$

The modular form $\Delta \in MF_{12}$ is not a permanent cycle in the descent spectral sequence for unlocalized Tmf, but Δ^{24} is. It turns out that the map

$$\pi_*\mathrm{tmf} \to \pi_*\mathrm{tmf}[\Delta^{-24}]$$

is injective. Motivated by this, we define the *periodic* TMF*-spectrum*[4] by

$$\mathrm{TMF} := \mathrm{tmf}[\Delta^{-24}].$$

This spectrum is $|\Delta^{24}| = 576$-periodic. We have

$$\mathrm{TMF} \simeq \mathrm{Tmf}[\Delta^{-24}] \simeq \mathcal{O}^{top}(\mathcal{M}_{ell})$$

where the last equivalence comes from the fact that $\mathcal{M}_{ell}$ is the complement of the zero-locus of Δ in $\overline{\mathcal{M}}_{ell}$.

Another variant comes from the consideration of level structures. Given a congruence subgroup $\Gamma \leq SL_2(\mathbb{Z})$, one can consider the modular forms of level Γ to be those holomorphic functions on the upper half plane which satisfy (1.3.1) for all $A \in \Gamma$, and which satisfy a holomorphicity condition at all of the cusps of the quotient

$$\mathcal{M}_{ell}(\Gamma)(\mathbb{C}) = \mathcal{H}//\Gamma.$$

Integral versions of $\mathcal{M}_{ell}(\Gamma)$ can be defined by considering moduli spaces of elliptic curves with certain types of *level structures*. The most common Γ which are considered are:

$$\Gamma_0(N) := \left\{ A \in SL_2(\mathbb{Z}) \ : \ A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\},$$
$$\Gamma_1(N) := \left\{ A \in SL_2(\mathbb{Z}) \ : \ A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\},$$
$$\Gamma(N) := \left\{ A \in SL_2(\mathbb{Z}) \ : \ A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

The corresponding moduli stacks $\mathcal{M}_{ell}(\Gamma_0(N))$ and $\mathcal{M}_{ell}(\Gamma_1(N))$ (respectively $\mathcal{M}_{ell}(\Gamma(N))$) are defined over $\mathbb{Z}[1/N]$ (respectively $\mathbb{Z}[1/N, \zeta_N]$), with R-points consisting of the groupoid of pairs

$$(C, \eta)$$

[4]We warn the reader that the connective cover of TMF is *not* tmf.

where C is an elliptic curve over R, and

$$\eta = \begin{cases} \text{a cyclic subgroup of } C \text{ of order } N, & \Gamma = \Gamma_0(N), \\ \text{a point of } C \text{ of exact order } N, & \Gamma = \Gamma_1(N), \\ \text{an isomorphism } C[N] \cong \mathbb{Z}/N \times \mathbb{Z}/N, & \Gamma = \Gamma(N). \end{cases}$$

In each of these cases, forgetting the level structure results in an étale map of stacks

$$\mathcal{M}_{ell}(\Gamma) \to \mathcal{M}_{ell} \tag{1.3.12}$$

and we define the associated (periodic) spectra of *topological modular forms, with level structure* by

$$\begin{aligned} \mathrm{TMF}_0(N) &:= \mathcal{O}^{top}(\mathcal{M}_{ell}(\Gamma_0(N))), \\ \mathrm{TMF}_1(N) &:= \mathcal{O}^{top}(\mathcal{M}_{ell}(\Gamma_1(N))), \\ \mathrm{TMF}(N) &:= \mathcal{O}^{top}(\mathcal{M}_{ell}(\Gamma(N))). \end{aligned}$$

Compactifications $\overline{\mathcal{M}}_{ell}(\Gamma)$ of the moduli stacks $\mathcal{M}_{ell}(\Gamma)$ above were constructed by Deligne and Rapoport [DR73]. The extensions of the maps (1.3.12) to these compactifications

$$\overline{\mathcal{M}}_{ell}(\Gamma) \to \overline{\mathcal{M}}_{ell}$$

are not étale, but they are log-étale. Hill and Lawson have shown that the sheaf $\mathcal{O}^{top}$ extends to the log-étale site of $\overline{\mathcal{M}}_{ell}$ [HL16], allowing us to define corresponding Tmf-spectra by

$$\begin{aligned} \mathrm{Tmf}_0(N) &:= \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}(\Gamma_0(N))), \\ \mathrm{Tmf}_1(N) &:= \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}(\Gamma_1(N))), \\ \mathrm{Tmf}(N) &:= \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}(\Gamma(N))). \end{aligned}$$

1.4 Homotopy groups of TMF at the primes 2 and 3

We now give an overview of the 2- and 3-primary homotopy groups of TMF. Detailed versions of these computations can be found in [Kon12], [Bau08], and some very nice charts depicting the answers were created by Henriques [Hen14]. The basic idea is to invoke the descent spectral sequence. The E_2-term is computed by imitating the argument of Proposition 1.3.7. The descent spectral sequence does not degenerate 2 or 3-locally, and differentials must be deduced using a variety of ad hoc methods similar to those used to compute differentials in the Adams-Novikov spectral sequence.

3-primary homotopy groups of TMF

Every elliptic curve over a $\mathbb{Z}_{(3)}$-algebra R can (upon taking a faithfully flat extension of R) be put in the form [Bau08]

$$C'_{a_2,a_4} : y^2 = 4x(x^2 + a_2 x + a_4), \quad a_i \in R$$

with

$$\Delta = a_4^2(16a_2^2 - 64a_4)$$

invertible. The isomorphisms of any such are of the form

$$\begin{aligned} f_{\lambda,r} : C'_{a_2,a_4} &\to C'_{a'_2,a'_4} \\ (x,y) &\mapsto (\lambda^2(x-r), \lambda^3 y) \end{aligned} \tag{1.4.1}$$

with

$$r^3 + a_2 r^2 + a_4 r = 0$$

and

$$\begin{aligned} a'_2 &= \lambda^2(a_2 + 3r), \\ a'_4 &= \lambda^4(a_4 + 2a_2 r + 3r^2). \end{aligned} \tag{1.4.2}$$

Following the template of the proof of Proposition 1.3.7, we observe that for the coordinate $z = x/y$ at ∞, we have

$$f^*_{\lambda,r} dz = \lambda^{-1} dz. \tag{1.4.3}$$

We may therefore use the λ factor to compute the sections of $\omega^{\otimes *}$.

Specifically, by setting $\lambda = 1$, we associate to this data a graded Hopf algebroid (A', Γ') with

$$\begin{aligned} A' &:= \mathbb{Z}_{(3)}[a_2, a_4, \Delta^{-1}], \quad |a_i| = i, \\ \Gamma' &:= A'[r]/(r^3 + a_2 r^2 + a_4 r), \quad |r| = 2 \end{aligned}$$

with right unit given by (1.4.2) (with $\lambda = 1$) and coproduct given by the composition of two isomorphisms of the form $f_{1,r}$.

Now consider the cover

$$U = \mathrm{Proj}(A') \to (\mathcal{M}_{ell})_{\mathbb{Z}_{(3)}}.$$

We deduce from (1.4.3) that

$$\omega^{\otimes k}(U) = A'_k$$

and more generally

$$\omega^{\otimes k}(U^{\times_{\mathcal{M}_{ell}}(s+1)}) = \left((\Gamma')^{\otimes_{A'} s}\right)_k .$$

It follows that the Čech complex for $\bigoplus_k \omega^{\otimes k}$ associated to the cover U is the cobar complex for the graded Hopf algebroid (A', Γ'). We deduce that the E_2-term of the descent spectral sequence is given by the cohomology of the Hopf algebroid (A', Γ'):

$$H^s((\mathcal{M}_{ell})_{\mathbb{Z}_{(3)}}, \omega^{\otimes k}) = H^{s,k}(A', \Gamma').$$

One computes (see [Bau08]):

Proposition 1.4.4.

$$H^{s,k}(A', \Gamma') = \frac{\mathbb{Z}_{(3)}[c_4, c_6, \Delta^{\pm}][\beta] \otimes E[\alpha]}{3\alpha, 3\beta, \alpha c_4, \alpha c_6, \beta c_4, \beta c_6, c_6^2 = c_4^3 - 1728\Delta}$$

where β is given by the Massey product

$$\beta = \langle \alpha, \alpha, \alpha \rangle$$

and the generators are in bidegrees (s, k):

$$\begin{aligned} |c_i| &= (0, i), & |\Delta| &= (0, 12) \\ |\alpha| &= (1, 2), & |\beta| &= (2, 6). \end{aligned}$$

Figure 1.1 displays the descent spectral sequence

$$H^{s,k}(A', \Gamma') \Rightarrow \pi_{2k-s}\mathrm{TMF}_{(3)}.$$

Here:

- Boxes correspond to $\mathbb{Z}_{(3)}$'s.
- Dots correspond to $\mathbb{Z}/3$'s.
- Lines of slope $1/3$ correspond to multiplication by α.
- Lines of slope $1/7$ correspond to the Massey product $\langle -, \alpha, \alpha \rangle$.
- Lines of slope $-r$ correspond to d_r-differentials.
- Dashed lines correspond to hidden α extensions.

We omit the factors coming from negative powers of Δ. In other words, the descent spectral sequence for TMF is obtained from Figure 1.1 by inverting Δ. The differential on $\beta\Delta$ comes from the Toda differential in the Adams-Novikov spectral sequence for the sphere, and this implies all of the other differentials. As the figure indicates, Δ^3 is a permanent cycle, and so $\pi_*\mathrm{TMF}_{(3)}$ is 72-periodic.

Under the Hurewicz homomorphism

$$\pi_* S_{(3)} \to \pi_* \mathrm{TMF}_{(3)}$$

the elements α_1 and β_1 map to α and β, respectively.

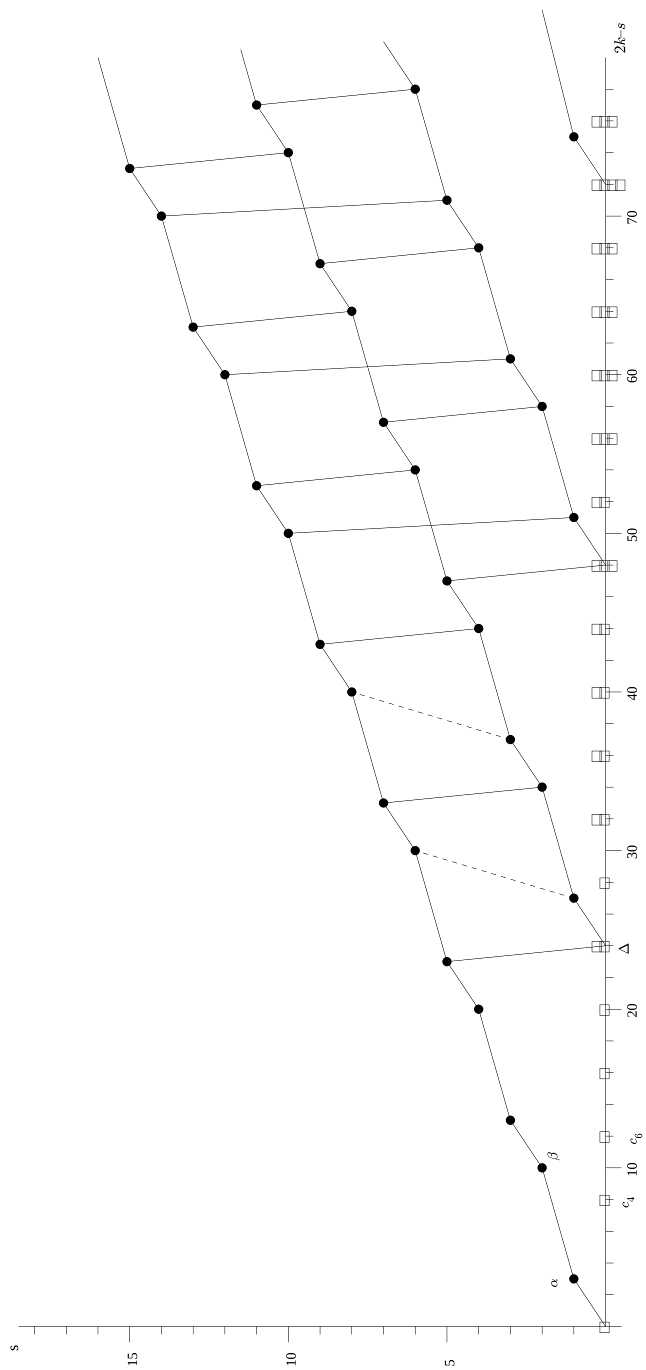

FIGURE 1.1
The descent spectral sequence for $\mathrm{tmf}_{(3)}$. The descent spectral sequence for $\mathrm{TMF}_{(3)}$ is obtained by inverting Δ.

2-primary homotopy groups of TMF

The analysis of the 2-primary descent spectral sequence proceeds in a similar fashion, except that the computations are significantly more involved. We will content ourselves to summarize the set-up, and then state the resulting homotopy groups of TMF, referring the reader to [Bau08] for the details.

Every elliptic curve over a $\mathbb{Z}_{(2)}$-algebra R can (upon taking an étale extension of R) be put in the form [Bau08]

$$C''_{a_1,a_3} : y^2 + a_1xy + a_3y = x^3, \quad a_i \in R$$

with

$$\Delta = a_3^3(a_1^3 - 27a_3)$$

invertible.

The isomorphisms of any such are of the form

$$\begin{aligned} f_{\lambda,s,t} : C''_{a_1,a_3} &\to C''_{a'_1,a'_3} \\ (x,y) &\mapsto (\lambda^2(x - 1/3(s^2 + a_1s)), \lambda^3(y - sx + 1/3(s^3 + a_1s^2) - t)) \end{aligned} \tag{1.4.5}$$

with[5]

$$s^4 - 6st + a_1s^3 - 3a_1t - 3a_3s = 0, \tag{1.4.6}$$

$$-27t^2 + 18s^3t + 18a_1s^2t - 27a_3t - 2s^6 - 3a_1s^5 + 9a_3s^3 + a_1^3s^3 + 9a_1a_3s^2 = 0, \tag{1.4.7}$$

and

$$\begin{aligned} a'_1 &:= \lambda(a_1 + 2s), \\ a'_3 &:= \lambda^3(a_3 + 1/3(a_1s^2 + a_1s) + 2t). \end{aligned} \tag{1.4.8}$$

Again, setting $\lambda = 1$, we associate to this data a graded Hopf algebroid (A'', Γ'') with

$$\begin{aligned} A'' &:= \mathbb{Z}_{(2)}[a_1, a_3, \Delta^{-1}], \quad |a_i| = i, \\ \Gamma'' &:= A''[s,t]/\sim, \quad |s| = 1,\ |t| = 3 \end{aligned}$$

(where $\sim$ consists of relations (1.4.6), (1.4.7)) with right unit given by (1.4.2) (with $\lambda = 1$) and coproduct given by the composition of two isomorphisms of the form $f_{1,s,t}$. The E_2-term of the descent spectral sequence takes the form

$$H^s((\mathcal{M}_{ell})_{(2)}, \omega^{\otimes k}) = H^{s,k}(A'', \Gamma'').$$

[5] We warn the reader that there may be a typo in the analog of (1.4.7) which appears in [Bau08, Sec. 7], as even using (1.4.6), relation (1.4.7) seems to be inconsistent with what appears there.

Proposition 1.4.9 ([Bau08], [Rez07]). *The cohomology of the Hopf algebroid (A'', Γ'') is given by*

$$H^{*,*}(A'', \Gamma'') = \mathbb{Z}_{(2)}[c_4, c_6, \Delta^{\pm}, \eta, a_1^2\eta, \nu, \epsilon, \kappa, \bar{\kappa}]/(\sim)$$

where $\sim$ consists of the relations

$$\begin{gathered} 2\eta,\ \eta\nu,\ 4\nu,\ 2\nu^2,\ \nu^3 = \eta\epsilon, \\ 2\epsilon,\ \nu\epsilon,\ \epsilon^2,\ 2a_1^2\eta,\ \nu a_1^2\eta,\ \epsilon a_1^2\eta,\ (a_1^2\eta)^2 = c_4\eta^2, \\ 2\kappa,\ \eta^2\kappa,\ \nu^2\kappa = 4\bar{\kappa},\ \epsilon\kappa,\ \kappa^2,\ \kappa a_1^2\eta, \\ \nu c_4,\ \nu c_6,\ \epsilon c_4,\ \epsilon c_6,\ a_1^2\eta c_4 = \eta c_6,\ a_1^2\eta c_6 = \eta c_4^2, \\ \kappa c_4,\ \kappa c_6,\ \bar{\kappa} c_4 = \eta^4\Delta,\ \bar{\kappa} c_6 = \eta^3(a_1^2\eta)\Delta,\ c_6^2 = c_4^3 - 1728\Delta \end{gathered}$$

and the generators are in bidegrees (s, k):

$$\begin{array}{lll} |c_i| = (0, i), & |\Delta| = (0, 12), & |\eta| = (1, 1), \\ |a_1^2\eta| = (1, 3), & |\nu| = (1, 2), & |\epsilon| = (2, 5), \\ |\kappa| = (2, 8), & |\bar{\kappa}| = (4, 12). & \end{array}$$

There are many differentials in the descent spectral sequence

$$H^{s,k}(A'', \Gamma'') \Rightarrow \pi_{2k-s}\mathrm{TMF}_{(2)}.$$

These were first determined by Hopkins, and first appeared in the preprint "From elliptic curves to homotopy theory" by Hopkins and Mahowald [HM14a], and we refer the reader to that paper or [Bau08] for the details.

We content ourselves with simply stating the resulting homotopy groups of $\mathrm{TMF}_{(2)}$. These are displayed in Figure 1.2. Our choice of names for elements in the descent spectral sequence (and our abusive practice of giving the elements of $\pi_*\mathrm{TMF}$ they detect the same names) is motivated by the fact that the elements

$$\eta, \nu, \epsilon, \kappa, \bar{\kappa}, q, u, w$$

in the 2-primary stable homotopy groups of spheres map to the corresponding elements in $\pi_*\mathrm{TMF}_{(2)}$. We warn the reader that there are many hidden extensions in the descent spectral sequence, so that often the names of elements in Figure 1.2 do not reflect the element which detects them in the descent spectral sequence because in the descent spectral sequence the product would be zero. For example, κ^2 is zero in $H^{*,*}(A'', \Gamma'')$, but nonzero in $\pi_*\mathrm{TMF}$. More complete multiplicative information can be found in [Hen14].

In Figure 1.2:

- A series of i black dots joined by vertical lines corresponds to a factor of $\mathbb{Z}/2^i$ which is annihilated by some power of c_4.
- An open circle corresponds to a factor of $\mathbb{Z}/2$ which is not annihilated by a power of c_4.

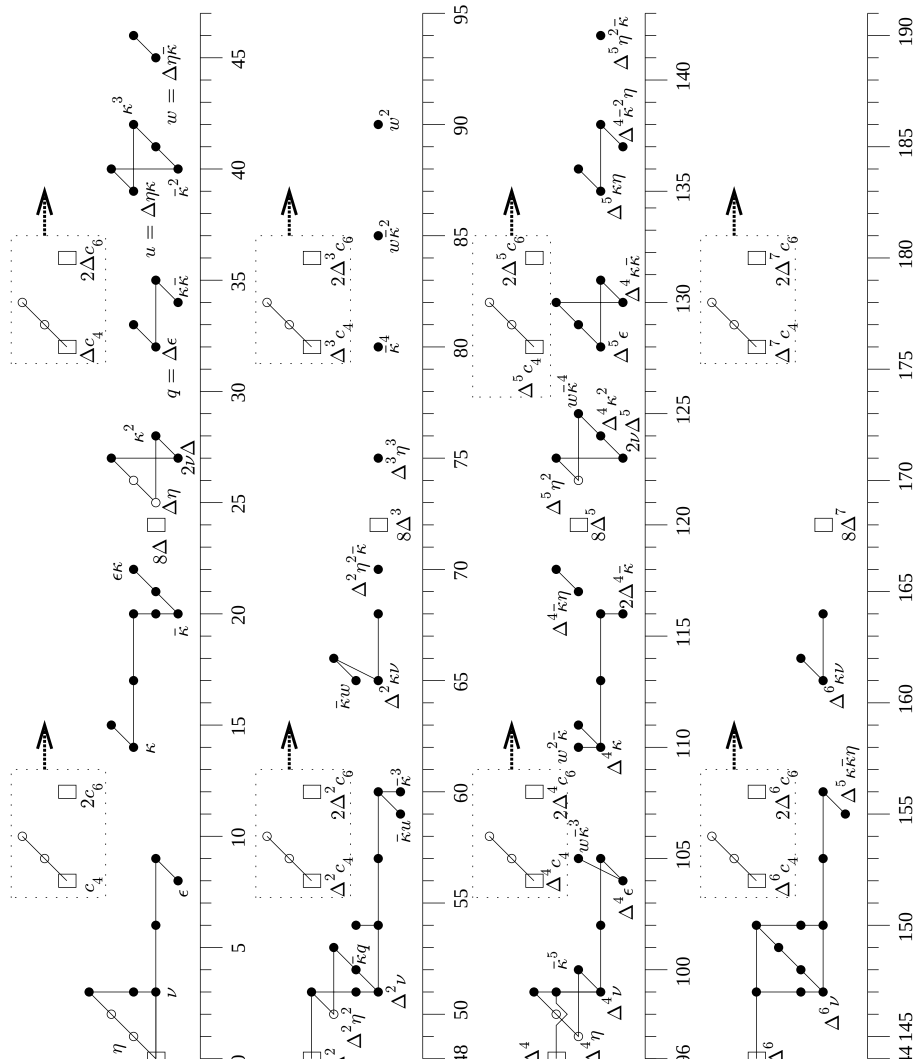

FIGURE 1.2
The homotopy groups of $\mathrm{tmf}_{(2)}$. $\pi_*\mathrm{TMF}_{(2)}$ is obtained by inverting Δ.

- A box indicates a factor of $\mathbb{Z}_{(2)}$ which is not annihilated by a power of c_4.
- The non-vertical lines indicate multiplication by η and ν.
- A pattern with a dotted box around it and an arrow emanating from the right face indicates this pattern continues indefinitely to the right by c_4-multiplication (i.e. tensor the pattern with $\mathbb{Z}_{(2)}[c_4]$).

The element Δ^8 is a permanent cycle, and $\pi_*\mathrm{TMF}_{(2)}$ is 192-periodic on the pattern depicted in Figure 1.2. The figure does not depict powers of c_4 supported by negative powers of Δ.

1.5 The homotopy groups of Tmf and tmf

We give a brief discussion of how the analysis in Section 1.4 can be augmented to determine $\pi_*\mathrm{Tmf}$, and thus $\pi_*\mathrm{tmf}$. We refer the reader to [Kon12] for more details. We have already described $\pi_*\mathrm{Tmf}[1/6]$ in Section 1.3, so we focus on $\pi_*\mathrm{Tmf}_{(p)}$ for $p = 2, 3$.

The ordinary locus

We first must describe a cover of $\overline{\mathcal{M}}_{ell}$. We recall that for a formal group F, the *p-series* is the formal power series

$$[p]_F(x) = \underbrace{x +_F \cdots +_F x}_{p}$$

where $x +_F y := F(x, y)$. If F is defined over a ring of characteristic p, we say it has *height n* if its p-series takes the form

$$[p]_F(x) = v_n^F x^{p^n} + \cdots$$

with v_n^F a unit.

Elliptic curves (over fields of characteristic p) have formal groups of height 1 or 2. We shall call a trivializable elliptic curve over a $\mathbb{Z}_{(p)}$-algebra R *ordinary* if the formal group $\widehat{\bar{C}}$ has height 1 (where $\bar{C}$ the base change of the curve to R/p). Let $(\mathcal{M}_{ell}^{ord})_{\mathbb{Z}_{(p)}}$ denote the moduli stack of ordinary elliptic curves, and define $(\overline{\mathcal{M}}_{ell}^{ord})_{\mathbb{Z}_{(p)}}$ to be the closure of $(\mathcal{M}_{ell}^{ord})_{\mathbb{Z}_{(p)}}$ in $(\overline{\mathcal{M}}_{ell})_{\mathbb{Z}_{(p)}}$.

We have the following lemma ([Rez07, Sec. 21]).

Lemma 1.5.1. *Let*

$$F' = \widehat{\bar{C}'}_{a_2,a_4},$$
$$F'' = \widehat{\bar{C}''}_{a_1,a_3},$$

denote the formal group laws of the reductions of the elliptic curves C'_{a_2,a_4} and C''_{a_1,a_3} modulo 3 and 2, respectively. Then we have[6]

$$v_1^{F'} = -a_2,$$
$$v_1^{F''} = a_1.$$

Define

$$\mathrm{TMF}^{ord} := \mathcal{O}^{top}((\mathcal{M}_{ell}^{ord})_{\mathbb{Z}_{(p)}}).$$

Using the fact that [Rez07, Prop.18.7]

$$c_4 \equiv \begin{cases} 16a_2^2 \pmod 3, & p = 3, \\ a_1^4 \pmod 2, & p = 2 \end{cases} \tag{1.5.2}$$

we have (for $p = 2$ or 3)

$$\mathrm{TMF}^{ord}_{(p)} = \mathrm{TMF}_{(p)}[c_4^{-1}].$$

We deduce from the computations of $\pi_*\mathrm{TMF}_{(p)}$:

Proposition 1.5.3. *We have*

$$\pi_*\mathrm{TMF}^{ord}_{(p)} = \begin{cases} \frac{\mathbb{Z}_{(3)}[c_4^{\pm}, c_6, \Delta^{\pm}]}{c_6^2 = c_4^3 - 1728\Delta}, & p = 3, \\ \frac{\mathbb{Z}_{(2)}[c_4^{\pm}, 2c_6, \Delta^{\pm}, \eta]}{2\eta, \eta^3, \eta\cdot(2c_6), (2c_6)^2 = 4(c_4^3 - 1728\Delta)}, & p = 2. \end{cases}$$

Using the covers

$$\mathrm{Proj}(\mathbb{Z}_{(3)}[a_2^{\pm}, a_4]) \to (\overline{\mathcal{M}}_{ell}^{ord})_{(3)},$$
$$\mathrm{Proj}(\mathbb{Z}_{(2)}[a_1^{\pm}, a_3]) \to (\overline{\mathcal{M}}_{ell}^{ord})_{(2)},$$

the Hopf algebroids (A', Γ') and (A'', Γ'') have variants where a_2 (respectively a_1) is inverted and Δ is not. Using these, one computes the descent spectral sequence for $\mathrm{Tmf}^{ord}_{(p)}$ at $p = 2, 3$ and finds:

Proposition 1.5.4. *We have:*

$$\pi_*\mathrm{Tmf}^{ord}_{(p)} = \begin{cases} \frac{\mathbb{Z}_{(3)}[c_4^{\pm}, c_6, \Delta]}{c_6^2 = c_4^3 - 1728\Delta}, & p = 3, \\ \frac{\mathbb{Z}_{(2)}[c_4^{\pm}, 2c_6, \Delta, \eta]}{2\eta, \eta^3, \eta\cdot(2c_6), (2c_6)^2 = 4(c_4^3 - 1728\Delta)}, & p = 2. \end{cases}$$

[6] For an elliptic curve C over a ring of characteristic p, $v_1^{\widehat{C}}$ is known as the *Hasse invariant*.

A homotopy pullback for $\mathrm{Tmf}_{(p)}$

The spectrum Tmf can be accessed at the primes 2 and 3 in a manner analogous to the case of Proposition 1.3.7: associated to the cover[7]

$$\left\{(\overline{\mathcal{M}}_{ell}^{ord})_{\mathbb{Z}_{(p)}}, (\mathcal{M}_{ell})_{\mathbb{Z}_{(p)}}\right\} \to (\overline{\mathcal{M}}_{ell})_{\mathbb{Z}_{(p)}}$$

there is a homotopy pullback (coming from the sheaf condition of $\mathcal{O}^{top}$)

$$\begin{tikzcd} \mathrm{Tmf}_{(p)} \arrow[r] \arrow[d] & \mathcal{O}^{top}((\mathcal{M}_{ell})_{\mathbb{Z}_{(p)}}) \arrow[d] \\ \mathcal{O}^{top}((\overline{\mathcal{M}}_{ell}^{ord})_{\mathbb{Z}_{(p)}}) \arrow[r] & \mathcal{O}^{top}((\mathcal{M}_{ell}^{ord})_{\mathbb{Z}_{(p)}}). \end{tikzcd} \tag{1.5.5}$$

Since we have described the homotopy groups of the spectra

$$\begin{aligned} \mathrm{TMF}_{(p)} &:= \mathcal{O}^{top}((\mathcal{M}_{ell})_{\mathbb{Z}_{(p)}}), \\ \mathrm{TMF}_{(p)}^{ord} &:= \mathcal{O}^{top}((\mathcal{M}_{ell}^{ord})_{\mathbb{Z}_{(p)}}), \\ \mathrm{Tmf}_{(p)}^{ord} &:= \mathcal{O}^{top}((\overline{\mathcal{M}}_{ell}^{ord})_{\mathbb{Z}_{(p)}}) \end{aligned}$$

at the primes 2 and 3, the homotopy groups of $\mathrm{Tmf}_{(p)}$ at these primes may be computed using the pullback square (1.5.5).

The homotopy groups of $\mathrm{tmf}_{(p)}$

Once one computes $\pi_*\mathrm{Tmf}_{(p)}$ it is a simple matter to read off the homotopy groups of the connective cover $\mathrm{tmf}_{(p)}$. We obtain:

Theorem 1.5.6. *The homotopy groups of* $\mathrm{tmf}_{(3)}$ *are given by the* E_∞*-page of the spectral sequence of Figure 1.1,*[8] *and the homotopy groups of* $\mathrm{tmf}_{(2)}$ *are depicted in Figure 1.2. These homotopy groups are* Δ^3 *(respectively* Δ^8*)-periodic.*

We end this section by stating a very useful folklore theorem which was proven rigorously in [Mat16].

Theorem 1.5.7 (Mathew)**.** *The mod* 2 *cohomology of* tmf *is given (as a module over the Steenrod algebra) by*

$$H^*(\mathrm{tmf}; \mathbb{F}_2) = A//A(2)$$

where A is the mod 2 Steenrod algebra, and $A(2)$ is the subalgebra generated by Sq^1, Sq^2, *and* Sq^4.

[7] The fact that this is a cover follows from the fact that $(\mathcal{M}_{ell}^{ord})_{\mathbb{Z}_{(p)}}$ is Zariski dense in $(\mathcal{M}_{ell})_{\mathbb{Z}_{(p)}}$.

[8] There are no additive extensions in this spectral sequence.

Corollary 1.5.8. *For a spectrum X, the Adams spectral sequence for the 2-adic* tmf*-homology of X takes the form*

$$\mathrm{Ext}^{s,t}_{A(2)}(H^*(X;\mathbb{F}_2),\mathbb{F}_2) \Rightarrow \pi_{t-s}(\mathrm{tmf} \wedge X)^\wedge_2.$$

1.6 Tmf from the chromatic perspective

We outline the essential algebro-geometric ideas behind chromatic homotopy theory, as originally envisioned by Morava [Mor85] (see also [Hop99], [DFHH14, Ch. 9], [Goe09]), and apply it to understand the chromatic localizations of Tmf. We will find that the pullback (1.5.5) used to access Tmf is closely related to its chromatic fracture square.

Stacks associated to ring spectra

The material in this section is closely aligned with that of Mike Hopkins's lecture "From spectra to stacks" in [DFHH14, Ch. 9]. In this lecture, Hopkins explains how non-complex orientable ring spectra give rise to stacks over the moduli stack of formal groups. This allows certain computations to be conceptualized by taking pullbacks over the moduli stack of formal groups.

The complex cobordism spectrum MU has a canonical complex orientation. To conform better to the even periodic set-up, we utilize the even periodic variant[9]

$$\mathrm{MUP} := \bigvee_{i\in\mathbb{Z}} \Sigma^{2i}\mathrm{MU}$$

so that $\pi_0\mathrm{MUP} \cong \pi_* MU$. Quillen proved [Qui69] (see also [Ada74]) that the associated formal group law F_{MUP} is the universal formal group law:

$$\mathrm{spec}(\pi_0\mathrm{MUP})(R) = \{\text{formal group laws over } R\}$$

In particular

$$\mathrm{spec}(\pi_0\mathrm{MUP}) \xrightarrow{F_{MUP}} \mathcal{M}_{fg}$$

is a flat cover. In fact, we have [Qui69], [Ada74]

$$\begin{aligned}\mathrm{spec}(\mathrm{MUP}_0\mathrm{MUP}) &= \{\text{isomorphisms } f : F \to F' \text{ between formal group laws over } R\} \\ &= \mathrm{spec}(\pi_0\mathrm{MUP}) \times_{\mathcal{M}_{fg}} \mathrm{spec}(\pi_0\mathrm{MUP}).\end{aligned}$$

Suppose that E is a complex oriented even periodic ring spectrum whose formal group law classifying map

$$\mathrm{spec}(\pi_0 E) \xrightarrow{F_E} \mathcal{M}_{fg}.$$

[9] Just as MU is the Thom spectrum of the universal virtual bundle over BU, MUP is the Thom spectrum of the universal virtual bundle over $BU \times \mathbb{Z}$.

is flat. We shall call such ring spectra *Landweber exact* (see Theorem 1.2.15). The formal group law F_E of such E determines E in the following sense: the classifying map

$$\pi_0\mathrm{MUP} \xrightarrow{F_E} \pi_0 E$$

lifts to a map of ring spectra

$$\mathrm{MUP} \to E$$

and the associated map

$$\pi_0 E \otimes_{\pi_0\mathrm{MUP}} \mathrm{MUP}_* X \to E_* X \tag{1.6.1}$$

is an isomorphism for all spectra X [Lan76], [Hop99]. Specializing to the case of $X = \mathrm{MUP}$, we have

$$\mathrm{MUP}_0 E \cong \pi_0 E \otimes_{\pi_0\mathrm{MUP}} \mathrm{MUP}_0\mathrm{MUP}$$

and hence the square

$$\begin{tikzcd}
\mathrm{spec}(\mathrm{MUP}_0 E) \arrow[r] \arrow[d] & \mathrm{spec}(\pi_0\mathrm{MUP}) \arrow[d, "F_{\mathrm{MUP}}"] \\
\mathrm{spec}(\pi_0 E) \arrow[r, "F_E"'] & \mathcal{M}_{fg}
\end{tikzcd}$$

is a pullback. We deduce that $\mathrm{MUP}_0 E$ is flat as an MUP_0-module.

We shall say that a commutative ring spectrum E is *even* if

$$\mathrm{MUP}_{\mathrm{odd}} E = 0.$$

We shall say that an even ring spectrum E is *Landweber* if $\mathrm{MUP}_0 E$ is flat as an MUP_0-module. Note by the previous paragraph every Landweber exact ring spectrum is Landweber. Since $\mathrm{MUP}_0 E$ is an $\mathrm{MUP}_0\mathrm{MUP}$-comodule algebra, the morphism

$$\mathrm{spec}(\mathrm{MUP}_0 E) \to \mathrm{spec}(\mathrm{MUP}_0)$$

comes equipped with descent data to determine a stack $\mathcal{X}_E$ and a flat morphism

$$\mathcal{X}_E \to \mathcal{M}_{fg}. \tag{1.6.2}$$

We shall call $\mathcal{X}_E$ the *stack associated to* E. Let ω denote the line bundle over $\mathcal{M}_{fg}$ whose fiber over a formal group law F is the cotangent space at the identity

$$\omega_F = T_0^* F.$$

We abusively also let ω denote the pullback of this line bundle to $\mathcal{X}_E$ under (1.6.2). Then an analysis similar to that of Section 1.4 (see [Dev18]) shows that the spectral sequence associated to the canonical Adams-Novikov resolution

$$E^{\wedge}_{MUP} := \mathrm{Tot}\left(\mathrm{MUP} \wedge E \rightrightarrows \mathrm{MUP} \wedge \mathrm{MUP} \wedge E \Rrightarrow \cdots\right)$$

takes the form

$$H^s(\mathcal{X}_E, \omega^{\otimes k}) \Rightarrow \pi_{2k-s} E^{\wedge}_{MUP}.$$

Example 1.6.3. The spectrum TMF is Landweber [Rez07, Sec. 20], [Mat16, Sec. 5.1], with

$$\mathcal{X}_{\mathrm{TMF}} = \mathcal{M}_{ell}.$$

The Adams-Novikov spectral sequence is the descent spectral sequence [HM14a]. In fact, the computations of [Rez07, Sec. 20] also show that tmf is Landweber, with

$$\mathcal{X}_{\mathrm{tmf}} = \mathcal{M}_{\mathrm{weier}}.$$

This is the moduli stack of *Weierstrass curves*, curves which locally take the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The associated Adams-Novikov spectral sequence

$$H^s(\mathcal{M}_{weier}, \omega^{\otimes k}) \Rightarrow \pi_{2k-s}\mathrm{tmf}$$

is computed in [Bau08]. The spectra $\mathrm{Tmf}^{ord}_{(p)}$ and $\mathrm{TMF}^{ord}_{(p)}$ are also Landweber, with

$$\mathcal{X}_{\mathrm{Tmf}^{ord}_{(p)}} = (\overline{\mathcal{M}}^{ord}_{ell})_{\mathbb{Z}_{(p)}}.$$
$$\mathcal{X}_{\mathrm{TMF}^{ord}_{(p)}} = (\mathcal{M}^{ord}_{ell})_{\mathbb{Z}_{(p)}}.$$

Unfortunately, the spectrum Tmf is not Landweber, but the pullback (1.5.5) does exhibit it as a pullback of Landweber ring spectra. The pushout of the corresponding diagram of stacks

$$\begin{tikzcd}
\mathcal{X}_{\mathrm{TMF}^{ord}_{(p)}} \arrow[r] \arrow[d] & \mathcal{X}_{\mathrm{TMF}_{(p)}} \arrow[d] \\
\mathcal{X}_{\mathrm{Tmf}^{ord}_{(p)}} \arrow[r] & (\overline{\mathcal{M}}_{ell})_{(\mathbb{Z}_{(p)})}
\end{tikzcd}$$

motivates us to consider $(\overline{\mathcal{M}}_{ell})_{(\mathbb{Z}_{(p)})}$ as the appropriate stack $\mathcal{X}_{\mathrm{Tmf}_{(p)}}$ over $\mathcal{M}_{fg}$ to associate to $\mathrm{Tmf}_{(p)}$. This motivates the following definition.

Definition 1.6.4. We shall call a ring spectrum E *locally Landweber* if it is given as a homotopy limit

$$E = \mathrm{holim}_{i \in \mathcal{I}} E_i$$

of Landweber ring spectra where $\mathcal{I}$ is a category whose nerve has finitely many non-degenerate simplices.[10] The colimit

$$\mathcal{X}_E := \mathrm{colim}_i\, \mathcal{X}_{E_i}$$

is the *stack associated to E*.

[10] We specify this condition so that homotopy limits taken over $\mathcal{I}$ commute with homotopy colimits in the category of spectra — see the proof of Proposition 1.6.6.

Remark 1.6.5. The stack $\mathcal{X}_E$ in the above definition a priori seems to to depend on the diagram $\{E_i\}$. In general, the E_2-term of the Adams-Novikov spectral sequence for E is *not* isomorphic to $H^s(\mathcal{X}_E, \omega^{\otimes k})$ (as happens in the case of Landweber spectra).

Proposition 1.6.6. *Suppose that E and E' are locally Landweber. Then so is $E \wedge E'$, and*

$$\mathcal{X}_{E\wedge E'} \simeq \mathcal{X}_E \times_{\mathcal{M}_{fg}} \mathcal{X}_{E'}.$$

Proof. Suppose first that E and E' are Landweber. The result then follows from the fact that the Landweber hypothesis yields a Künneth isomorphism

$$\mathrm{MUP}_0(E \wedge E') \cong \mathrm{MUP}_0(E) \otimes_{\mathrm{MUP}_0} \mathrm{MUP}_0(E').$$

Now suppose that E and E' are locally Landweber, given as limits

$$\begin{aligned} E &\simeq \operatorname*{holim}_{i \in \mathcal{I}} E_i, \\ E' &\simeq \operatorname*{holim}_{j \in \mathcal{J}} E'_j. \end{aligned}$$

Then the finiteness conditions on $\mathcal{I}$ and $\mathcal{J}$ allow us to compute

$$\begin{aligned} \operatorname*{holim}_{i,j}(E_i \wedge E'_j) &\simeq (\operatorname*{holim}_{i} E_i) \wedge (\operatorname*{holim}_{j} E'_j) \\ &\simeq E \wedge E'_j \end{aligned}$$

and we have

$$\begin{aligned} \mathcal{X}_E \times_{\mathcal{M}_{fg}} \mathcal{X}_{E'} =& (\operatorname{colim}_i \mathcal{X}_{E_i}) \times_{\mathcal{M}_{fg}} (\operatorname{colim}_j \mathcal{X}_{E_j}) \\ &= \operatorname{colim}_{i,j} \mathcal{X}_{E_i} \times_{\mathcal{M}_{fg}} \mathcal{X}_{E_j} \\ &\simeq \operatorname{colim}_{i,j} \mathcal{X}_{E_i \wedge E'_j} \\ &= \mathcal{X}_{E \wedge E'}. \end{aligned}$$

□

The stacks associated to chromatic localizations

Let

$$(\mathcal{M}_{fg})^{\le n}_{\mathbb{Z}_{(p)}} \subset \mathcal{M}_{fg}$$

denote the substack which classifies p-local formal group laws of height $\le n$. Let

$$(\mathcal{M}_{fg})^{[n]}_{\mathbb{Z}_{(p)}} \subset (\mathcal{M}_{fg})^{\le n}_{\mathbb{Z}_{(p)}}$$

denote the formal neighborhood[11] of the locus of formal group laws in characteristic p of exact height n.

[11] $(\mathcal{M}_{fg})^{[n]}_{\mathbb{Z}_{(p)}}$ is technically a *formal stack*.

Over $\bar{\mathbb{F}}_p$, any two formal groups of height n are isomorphic. Lubin and Tate showed that given a height n formal group F over $\mathbb{F}_q$ (a finite extension of $\mathbb{F}_p$), its universal deformation space is a formally affine Galois cover

$$\mathrm{Spf}(\mathbb{Z}_p[\zeta_{q-1}][[u_1, \dots, u_{n-1}]]) = \mathcal{X}_n^F \to (\mathcal{M}_{fg})^{[n]}_{\mathbb{Z}_{(p)}}$$

If the formal group F is defined over $\mathbb{F}_p$, and q is sufficiently large, the profinite Galois group takes the form

$$\mathbb{G}_n^F = \mathbb{S}_n^F \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$$

where $\mathbb{S}_n^F = \mathrm{aut}(F)$ is the associated *Morava stabilizer group*.[12]

Fix a prime p, and let $E(n)$ denote the nth Johnson-Wilson spectrum, $K(n)$ the nth Morava K-theory spectrum, and E_n^F the nth Morava E-theory spectrum (associated to a height n formal group F over a finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$),[13] [JW73], [JW75], [Mor78] with

$$\begin{aligned} \pi_* E(n) &= \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm}], \\ \pi_* K(n) &= \mathbb{F}_p[v_n^{\pm}], \\ \pi_* E_n^F &= \mathbb{Z}_p[\zeta_{q-1}][[u_1, \dots, u_{n-1}]][u^{\pm}]. \end{aligned}$$

Here, $|v_i| = 2(p^i - 1)$, $|u_i| = 0$, and $|u| = -2$. The spectra $E(n)$ and E_n^F are Landweber exact.

For spectra X and E, let X_E denote the E-localization of the spectrum X [Bou79]. Then we have the following correspondence between locally Landweber spectra and associated stacks over $\mathcal{M}_{fg}$:

spectrum E	stack $\mathcal{X}_E$
S	$\mathcal{M}_{fg}$
$S_{E(n)}$	$(\mathcal{M}_{fg})^{\le n}_{\mathbb{Z}_{(p)}}$
$S_{K(n)}$	$(\mathcal{M}_{fg})^{[n]}_{\mathbb{Z}_{(p)}}$
E_n^F	$\mathcal{X}_n^F$

Remark 1.6.7. The spectrum $E = S_{K(n)}$ is really only Landweber in the $K(n)$-local category, in the sense that $(\mathrm{MUP} \wedge E)_{K(n)}$ is Landweber exact.

[12] Often, the formal group F is taken to be the Honda height n formal group over $\mathbb{F}_{p^n}$, and the associated Morava stabilizer group is simply denoted $\mathbb{S}_n$.

[13] When the formal group F is the Honda height n formal group, the associated Morava E-theory spectrum is often denoted E_n.

However, similar considerations associate formal stacks $\mathcal{X}_E$ to such $K(n)$-local ring spectra, and an analog of Proposition 1.6.6 holds where

$$\mathcal{X}_{(E \wedge E')_{K(n)}} \simeq \mathcal{X}_E \widehat{\times}_{\mathcal{M}_{fg}} \mathcal{X}_{E'}.$$

The spectrum $S_{E(n)}$ is a limit of spectra which are Landweber in the above $K(i)$-local sense $i \leq n$ [ACB14].

Galois descent is encoded in the work of Goerss-Hopkins-Miller [GH04] and Devinatz-Hopkins [DH04], who showed that the group $\mathbb{G}_n^F$ acts on E_n^F with

$$S_{K(n)} \simeq (E_n^F)^{h\mathbb{G}_n^F}.$$

The following proposition follows from Proposition 1.6.6 (or its $K(n)$-local variant), and is closely related to localization forulas which appear in [GM95].

Proposition 1.6.8. *Suppose that E is locally Landweber. Then so is $E_{E(n)}$ and $E_{K(n)}$, and the associated stacks are given as the pullbacks*

$$\begin{tikzcd} \mathcal{X}_{E_{E(n)}} \arrow[r] \arrow[d] & (\mathcal{M}_{fg})^{\leq n}_{\mathbb{Z}_{(p)}} \arrow[d] \\ \mathcal{X}_E \arrow[r] & \mathcal{M}_{fg} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}_{E_{K(n)}} \arrow[r] \arrow[d] & (\mathcal{M}_{fg})^{[n]}_{\mathbb{Z}_{(p)}} \arrow[d] \\ \mathcal{X}_E \arrow[r] & \mathcal{M}_{fg} \end{tikzcd}$$

and

$$E_{K(n)} = (E_{E(n)})^{\wedge}_{I_n}$$

where $I_n = (p, v_1, \dots, v_{n-1})$ is the ideal corresponding to the locus of height n formal groups in $(\mathcal{M}_{fg})^{\leq n}_{\mathbb{Z}_{(p)}}$.

For a general spectrum X, the square

$$\begin{tikzcd} X_{E(n)} \arrow[r] \arrow[d] & X_{K(n)} \arrow[d] \\ X_{E(n-1)} \arrow[r] & (X_{K(n)})_{E(n-1)}. \end{tikzcd}$$

is a homotopy pullback (the *chromatic fracture square*). If X is a locally Landweber ring spectrum, the chromatic fracture square can be regarded as the being associated to the "cover"

$$\left\{ (\mathcal{M}_{fg})^{\leq n-1}_{\mathbb{Z}_{(p)}}, (\mathcal{M}_{fg})^{[n]}_{\mathbb{Z}_{(p)}} \right\} \to (\mathcal{M}_{fg})^{\leq n}_{\mathbb{Z}_{(p)}}.$$

$K(1)$-local Tmf

Applying Proposition 1.6.8, we find

$$\mathrm{Tmf}_{K(1)} \simeq \mathrm{tmf}_{K(1)} \simeq (\mathrm{Tmf}^{ord}_{(p)})^\wedge_p.$$

We explain the connection of $K(1)$-local Tmf to the Katz-Serre theory of p-adic modular forms.

The ring of *divided congruences* is defined to be the ring of inhomogeneous sums of rational modular forms where the sum of the q-expansions is integral:

$$D := \left\{ \sum f_k \in \bigoplus_k MF_k \otimes \mathbb{Q} \ : \ \sum f_k(\tau) \in \mathbb{Z}[[q]] \right\}.$$

This ring was studied extensively by Katz [Kat75], who showed in [Kat73] that there is an isomorphism

$$D \cong \Gamma \mathcal{O}_{\overline{\mathcal{M}}^{triv}_{ell}}$$

where $\overline{\mathcal{M}}^{triv}_{ell}$ is the pullback

$$\begin{array}{ccc} \overline{\mathcal{M}}^{triv}_{ell} & \longrightarrow & \overline{\mathcal{M}}_{ell} \\ \downarrow & & \downarrow \\ \mathrm{spec}(\mathbb{Z}) & \xrightarrow[\widehat{\mathbb{G}}_m]{} & \mathcal{M}_{fg} \end{array}$$

Since complex K-theory is the Landweber exact ring spectrum associated to $\widehat{\mathbb{G}}_m$, Proposition 1.6.6 recovers the following theorem of Laures [Lau99].

Proposition 1.6.9 (Laures)**.** *The complex K-theory of* Tmf *is given by*

$$K_0\mathrm{Tmf} \cong D.$$

The ring of *generalized p-adic modular functions* [Kat73], [Kat75] is the p-completion of the ring D:

$$V_p := D^\wedge_p$$

and the proposition above implies that there is an isomorphism

$$\pi_0(K \wedge \mathrm{Tmf})^\wedge_p \cong V_p.$$

For $p \not| \ell$, the action of the ℓth Adams operation ψ^ℓ on this space coincides with the action of $\ell \in \mathbb{Z}_p^\times$ on V described in [Kat75], and

$$V_p\langle k \rangle = \{ f \in V \ : \ \psi^\ell f = \ell^k f \}$$

is isomorphic to Serre's space of p-adic modular forms of weight k [Ser73].

Letting p be odd, and choosing ℓ to be a topological generator of $\mathbb{Z}_p^\times$, we deduce from the fiber sequence [HMS94, Lem. 2.5]

$$S^{2k}_{K(1)} \to K_p^\wedge \xrightarrow{\psi^\ell - \ell^k} K_p^\wedge$$

the following theorem of Baker [Bak89].

Proposition 1.6.10. *For $p \geq 3$, the homotopy groups of $\mathrm{Tmf}_{K(1)}$ are given by the spaces of p-adic modular forms*

$$\pi_{2k}\mathrm{Tmf}_{K(1)} \cong V_p\langle k \rangle.$$

Remark 1.6.11. At the prime $p = 2$, Hopkins [Hop14] and Laures [Lau04] studied the spectrum $\mathrm{Tmf}_{K(1)}$, and showed that it has a simple construction as a finite cell object in the category of $K(1)$-local E_∞-ring spectra.

$K(2)$-**local** Tmf

An elliptic curve C over a field of characteristic p is called *supersingular* if its formal group $\widehat{C}$ has height 2. Over $\bar{\mathbb{F}}_p$ there are only finitely many isomorphism classes of supersingular elliptic curves. We shall let $(\mathcal{M}^{ss}_{ell})_{\mathbb{Z}_{(p)}}$ denote the formal neighborhood of the supersingular locus in $(\mathcal{M}_{ell})_{\mathbb{Z}_{(p)}}$.

Serre-Tate theory [LS64] implies that deformations of the supersingular elliptic curve C are in bijective correspondence with the deformations of its formal group $\widehat{C}$. Therefore the Lubin-Tate space $\widetilde{X}_2^{\widehat{C}}$ carries the universal deformation $\widetilde{C}$ of the elliptic curve C, giving us a lift:

$$\begin{tikzcd}
& \mathcal{X}_2^{\widehat{C}} \arrow[dl, dotted, "\widetilde{C}"'] \arrow[d] \\
(\mathcal{M}^{ss}_{ell})_{\mathbb{Z}_{(p)}} \arrow[r] \arrow[d] & (\mathcal{M}_{fg})^{[2]}_{\mathbb{Z}_{(p)}} \arrow[d] \\
\overline{\mathcal{M}}_{ell} \arrow[r] & \mathcal{M}_{fg}
\end{tikzcd}$$

One may deduce from this that the square in the diagram above is a pullback. By Proposition 1.6.8, we have

$$\mathcal{X}_{\mathrm{Tmf}_{K(2)}} \simeq \mathcal{X}_{\mathrm{tmf}_{K(2)}} \simeq (\mathcal{M}^{ss}_{ell})_{\mathbb{Z}_{(p)}}.$$

For $p = 2, 3$ there is only one supersingular curve (defined over $\mathbb{F}_p$), and $\widetilde{C}$ is a Galois cover, with Galois group equal to

$$\mathrm{aut}(C) \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p) \leq \mathbb{G}_2^{\widehat{C}}$$

where

$$|\mathrm{aut}(C)| = \begin{cases} 24, & p=2, \\ 12, & p=3. \end{cases}$$

Since the Morava E-theory $E_2^{\widehat{C}}$ is Landweber exact with $\mathcal{X}_{E_2^{\widehat{C}}} = \mathcal{X}_2^{\widehat{C}}$, we have the following.

Proposition 1.6.12. *For $p = 2, 3$ there are equivalences*

$$\mathrm{Tmf}_{K(2)} \simeq \mathrm{TMF}_{K(2)} \simeq \mathrm{tmf}_{K(2)} \simeq (E_2^{\widehat{C}})^{h\,\mathrm{aut}(C) \rtimes Gal(\mathbb{F}_{p^2}/\mathbb{F}_p)}.$$

Remark 1.6.13. For a maximal finite subgroup $G \le \mathbb{G}_n^F$, the homotopy fixed point spectrum spectrum $(E_n^F)^{hG}$ is sometimes denoted EO_n. Therefore the proposition above is stating that at the primes 2 and 3 there is an equivalence

$$\mathrm{Tmf}_{K(2)} \simeq EO_2.$$

Finally, we note that combining Proposition 1.6.8 with Lemma 1.5.1 and (1.5.2) yields the following.

Proposition 1.6.14. *There is an isomorphism*

$$\pi_*(\mathrm{TMF}_{K(2)}) \cong \pi_*(\mathrm{TMF})^\wedge_{p,c_4}$$

for $p = 2, 3$.

Chromatic fracture of Tmf

Since every elliptic curve in characteristic p has height 1 or 2, we deduce that the square

$$\begin{array}{ccc} (\overline{\mathcal{M}}_{ell})_{\mathbb{Z}_{(p)}} & \longrightarrow & (\mathcal{M}_{fg})^{\le 2}_{\mathbb{Z}_{(p)}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{ell} & \longrightarrow & \mathcal{M}_{fg} \end{array}$$

is a pullback. We deduce from Proposition 1.6.8 that $\mathrm{Tmf}_{(p)}$ and $\mathrm{TMF}_{(p)}$ are $E(2)$-local.[14]

The p-completion of the chromatic fracture square for Tmf

$$\begin{array}{ccc} \mathrm{Tmf}_p^\wedge & \longrightarrow & \mathrm{Tmf}_{K(2)} \\ \downarrow & & \downarrow \\ \mathrm{Tmf}_{K(1)} & \longrightarrow & (\mathrm{Tmf}_{K(2)})_{K(1)} \end{array}$$

[14] The spectrum $\mathrm{tmf}_{(p)}$ is not $E(2)$-local, as cuspidal Weierstrass curves in characteristic p have formal groups of infinite height.

therefore takes the form

$$\begin{tikzcd}
\mathrm{Tmf}_p^\wedge \arrow[r] \arrow[d] & \mathrm{TMF}_{(p,c_4)}^\wedge \arrow[d] \\
(\mathrm{Tmf}^{ord})_p^\wedge \arrow[r] & (\mathrm{TMF}_{c_4}^\wedge[c_4^{-1}])_p^\wedge
\end{tikzcd} \tag{1.6.15}$$

and corresponds to the cover

$$\{(\mathcal{M}_{ell}^{ss})_p^\wedge, (\overline{\mathcal{M}}_{ell}^{ord})_p^\wedge\} \to (\overline{\mathcal{M}}_{ell})_p^\wedge.$$

The p-completed chromatic fracture square for Tmf is therefore a completed version of the homotopy pullback (1.5.5).

Remark 1.6.16. For an elliptic curve C in characteristic p for $p \geq 5$, we have

$$v_1^{\widehat{C}} = E_{p-1}(C)$$

where E_{p-1} is the normalized Eisenstein series of weight $p-1$. We therefore have analogs of Proposition 1.6.14 and (1.6.15) for $p \geq 5$ where we replace c_4 with the modular form E_{p-1}.

1.7 Topological automorphic forms

p-divisible groups

Fix a prime p.

Definition 1.7.1. A *p-divisible group of height n* over a ring R is a sequence of commutative group schemes

$$0 = G_0 \leq G_1 \leq G_2 \leq \cdots$$

so that each G_i is locally free of rank p^{in} over R, and such that for each i we have

$$G_i = \ker([p^i] : G_{i+1} \to G_{i+1})$$

Example 1.7.2. Suppose that A is an abelian variety over R of dimension n. Then the sequence of group schemes $A[p^\infty] := \{A[p^i]\}$ given by the p^i-torsion points of A

$$A[p^i] := \ker([p^i] : A \to A)$$

is a p-divisible group of height $2n$.

Example 1.7.3. Suppose that F is a formal group law over a p-complete ring R of height n. Then the sequence of group schemes $F[p^\infty] = \{F[p^i]\}$ where

$$F[p^i] := \mathrm{spec}(R[[x]]/([p^i]_F(x)))$$

is a p-divisible group of height n.

Given a p-divisible group $G = \{G_i\}$ of height n over a p-complete ring R, the formal neighborhood of the identity

$$\widehat{G} \hookrightarrow \mathrm{colim}_i\, G_i$$

is a formal group of height $\leq n$ [Mes72]. We define the *dimension* of G to be the dimension of the formal group $\widehat{G}$. We shall say G is *trivializable* if the line bundle $T_0^*\widehat{G}$ is trivial.

If A is an abelian variety of dimension n over R, then we have

$$\widehat{A[p^\infty]} = \widehat{A}$$

and the dimension of $A[p^\infty]$ is n.

Lurie's theorem

Let $\mathcal{M}^n_{pd}$ denote the moduli stack of 1-dimensional p-divisible groups of height n, and let $(\mathcal{M}^n_{pd})^{DM}_{et}$ denote the site of formally étale maps

$$\mathcal{X} \xrightarrow{G} \mathcal{M}^n_{pd} \tag{1.7.4}$$

where $\mathcal{X}$ is a locally Noetherian separated Deligne-Mumford stack over a complete local ring with perfect residue field of characteristic p.

Remark 1.7.5. One typically checks that a map (1.7.4) is formally étale by checking that for each closed point $x \in \mathcal{X}$, the formal neighborhood of x is isomorphic to the universal deformation space of the fiber G_x.

Lurie proved the following seminal theorem [Lur18a].

Theorem 1.7.6 (Lurie)**.** *There is a sheaf $\mathcal{O}^{top}$ of E_∞ ring spectra on $(\mathcal{M}^n_{pd})^{DM}_{et}$ with the following property: the ring spectrum*

$$E := \mathcal{O}^{top}\left(\mathrm{spec}(R) \xrightarrow{G} \mathcal{M}^n_{pd}\right)$$

(associated to an affine formal étale open with G trivializable) is even periodic, with

$$F_E \cong \widehat{G}.$$

This theorem generalizes the Goerss-Hopkins-Miller theorem [GH04]. Consider the Lubin-Tate universal deformation space

$$\mathcal{X}_n^F \xrightarrow{\widetilde{F}} \mathcal{M}_{fg}^{[n]}.$$

The map classifying the p-divisible group $\widetilde{F}[p^\infty]$

$$\mathcal{X}_n^F \xrightarrow{\widetilde{F}[p^\infty]} \mathcal{M}_{pd}^n$$

is formally étale, simply because the data of a formal group is the same thing as the data of its associated p-divisible group over a p-complete ring, so they have the same deformations (see Remark 1.7.5). The associated ring spectrum is Morava E-theory:

$$\mathcal{O}^{top}(\mathcal{X}_n^F) \simeq E_n^F.$$

The functoriality of $\mathcal{O}^{top}$ implies that $\mathbb{G}_n^F$ acts on E_n^F.

Theorem 1.7.6 also generalizes (most of) Theorem 1.3.2. Serre-Tate theory states that deformations of abelian varieties are in bijective correspondence with deformations of their p-divisible groups. Again using Remark 1.7.5, this implies that the map

$$\begin{aligned} (\mathcal{M}_{ell})_{\mathbb{Z}_p} &\to \mathcal{M}_{pd}^2, \\ C &\mapsto C[p^\infty] \end{aligned}$$

is formally étale. We deduce the existence of $\mathcal{O}^{top}$ on $(\mathcal{M}_{ell})_{\mathbb{Z}_p}$.

Cohomology theories associated to certain PEL Shimura stacks

The main issue which prevents us from associating cohomology theories to general n-dimensional abelian varieties is that their associated p-divisible groups are not 1-dimensional (unless $n = 1$, of course).

PEL Shimura stacks are moduli stacks of abelian varieties with the extra structure of Polarization, Endomorphisms, and Level structure [Shi00], [HT01], [Hid04]. We will now describe a class of PEL Shimura stacks (associated to a rational form of the unitary group $U(1, n-1)$) whose PEL data allow for the extraction of a 1-dimensional p-divisible group satisfying the hypotheses of Theorem 1.7.6.

In order to define our Shimura stack $\mathcal{X}_{V,L}$, we need to fix the following data.

$$
\begin{aligned}
F &= \text{quadratic imaginary extension of } \mathbb{Q}\text{, such that } p \text{ splits as } u\bar{u}. \\
\mathcal{O}_F &= \text{ring of integers of } F. \\
V &= F\text{-vector space of dimension } n. \\
\langle -,- \rangle &= \mathbb{Q}\text{-valued non-degenerate hermitian alternating form on } V \\
&\qquad (\text{i.e. } \langle \alpha x, y\rangle = \langle x, \bar{\alpha} y\rangle \text{ for } \alpha \in F) \text{ of signature } (1, n-1). \\
L &= \mathcal{O}_F\text{-lattice in } V\text{, } \langle -,- \rangle \text{ restricts to give integer values on } L, \\
&\qquad \text{and makes } L_{(p)} \text{ self-dual.}
\end{aligned}
$$

The group of complex linear isometries (with respect to the form $\langle -,- \rangle$) of $V \otimes \mathbb{R}$ turns out to be a unitary group — the signature above is the signature of this unitary group.

Assume that S is a locally Noetherian scheme on which p is locally nilpotent. The groupoid $\mathcal{X}_{V,L}(S)$ consist of tuples of data (A, i, λ), and isomorphisms of such, defined as follows.

A	is an abelian scheme over S of dimension n.
$\lambda : A \to A^\vee$	is a polarization (principle at p), with Rosati involution $\dagger$ on $\mathrm{End}(A)_{(p)}$.
$i : \mathcal{O}_{F,(p)} \hookrightarrow \mathrm{End}(A)_{(p)}$	is an inclusion of rings, satisfying $i(\bar{z}) = i(z)^\dagger$.

We impose the following two conditions (one at p, one away from p) which basically amount to saying that the tuple (A, i, λ) is locally modeled on $(L, \langle -,- \rangle)$:

1. The splitting $p = u\bar{u}$ in $\mathcal{O}_F$ induces a splitting $\mathcal{O}_{F,p} = \mathcal{O}_{F,u} \times \mathcal{O}_{F,\bar{u}}$, and hence a splitting of p-divisible groups
$$A[p^\infty] \cong A[u^\infty] \oplus A[\bar{u}^\infty].$$
We require that $A[u^\infty]$ is 1-dimensional.

2. Choose a geometric point s in each component of S. We require that for each of these points there *exists* an $\mathcal{O}_F$-linear integral uniformization
$$\eta : \prod_{\ell \neq p} L_\ell^\wedge \xrightarrow{\cong} \prod_{\ell \neq p} T_\ell(A_s)$$
(where $T_\ell A = \lim_i A[\ell^i]$ is the ℓ-adic Tate module) which, when tensored with $\mathbb{Q}$, sends $\langle -,- \rangle$ to an $(\mathbb{A}^{p,\infty})^\times$-multiple of the λ-Weil pairing.[15]

[15]Here, $\mathbb{A}^{p,\infty} := \left(\prod_{\ell \neq p} \mathbb{Z}_\ell\right) \otimes \mathbb{Q}$ are the *adeles away from p and ∞*.

Given a tuple $(A, i, \lambda) \in \mathcal{X}_{V,L}(S)$, the conditions on i and λ imply that the polarization induces an isomorphism

$$\lambda_* : A[u^\infty] \xrightarrow{\cong} A[\bar{u}^\infty]^\vee \tag{1.7.7}$$

(where $(-)^\vee$ denotes the Cartier dual). This implies that the p-divisible group $A[u^\infty]$ has height n. Serre-Tate theory [LS64] implies that deformations of an abelian variety are in bijective correspondence with the deformations of its p-divisible group. The isomorphism (1.7.7) therefore implies that deformations of a tuple (A, i, λ) are in bijective correspondence with deformations of $A[u^\infty]$. By Remark 1.7.5, the map

$$\mathcal{X}_{V,L} \to \mathcal{M}^n_{pd}$$

is therefore formally etale. Applying Lurie's theorem, we obtain

Theorem 1.7.8 ([BL10])**.** *There exists a sheaf $\mathcal{O}^{top}$ of E_∞ ring spectra on the site $(\mathcal{X}_{V,L})_{et}$, such that for each affine étale open*

$$\mathrm{spec}(R) \xrightarrow{(A,i,\lambda)} \mathcal{X}_{V,L}$$

with $A[u^\infty]$ trivializable, the associated ring spectrum

$$E := \mathcal{O}^{top}(\mathrm{spec}(R) \xrightarrow{(A,i,\lambda)} \mathcal{X}_{V,L})$$

is even periodic with

$$F_E \cong \widehat{A[u^\infty]}.$$

The spectrum of *topological automorphic forms* (TAF) for the Shimura stack $\mathcal{X}_{V,L}$ is defined to be the spectrum of global sections

$$\mathrm{TAF}_{V,L} := \mathcal{O}^{top}(\mathcal{X}_{V,L}).$$

Let ω be the line bundle over $\mathcal{X}_{V,L}$ with fibers given by

$$\omega_{A,i,\lambda} = T_0^* \widehat{A[u^\infty]}.$$

Then the construction of the descent spectral sequence (1.3.5) goes through verbatim to give a descent spectral sequence

$$E_2^{s,2k} = H^s(\mathcal{X}_{V,L}, \omega^{\otimes k}) \Rightarrow \pi_{2k-s}\mathrm{TAF}_{V,L}.$$

The motivation behind the terminology "topological automorphic forms" is that the space of sections

$$AF_k(U(V), L)_{\mathbb{Z}_p} := H^0(\mathcal{X}_{V,L}, \omega^{\otimes k})$$

is the space of scalar valued weakly holomorphic automorphic forms for the unitary group $U(V)$ (associated to the lattice L) of weight k over $\mathbb{Z}_p$.

Remark 1.7.9. As in the modular case, the space of holomorphic automorphic forms has an additional growth condition which is analogous to the requirement that a modular form be holomorphic at the cusp. The term "weakly holomorphic" means that we drop this requirement. However, for $n \geq 3$, it turns out that every weakly holomorphic automorphic form is holomorphic [Shi00, Sec. 5.2].

The spectra $\mathrm{TAF}_{V,L}$ are locally Landweber, with

$$\mathcal{X}_{\mathrm{TAF}_{V,L}} \simeq \mathcal{X}_{V,L}.$$

The height of the formal groups $\widehat{A[u^\infty]}$ associated to mod p points (A, i, λ) of the Shimura stack $\mathcal{X}_{V,L}$ range from 1 to n. We deduce from Proposition 1.6.8 that $\mathrm{TAF}_{V,L}$ is $E(n)$-local, and an analysis similar to that in the Tmf case (see Section 1.6) yields the following.

Proposition 1.7.10 ([BL10])**.** *The $K(n)$-localization of $\mathrm{TAF}_{V,L}$ is given by*

$$(\mathrm{TAF}_{V,L})_{K(n)} \simeq \left(\prod_{(A,i,\lambda)} \left(E_n^{\widehat{A[u^\infty]}} \right)^{h\,\mathrm{aut}(A,i,\lambda)} \right)^{h\mathrm{Gal}}$$

where the product ranges over the (finite, non-empty) set of mod p points (A, i, λ) of $\mathcal{X}_{V,L}$ with $\widehat{A[u^\infty]}$ of height n.

The groups $\mathrm{aut}(A, i, \lambda)$ are finite subgroups of the Morava stabilizer group. The structure of these subgroups, and the conditions under which they are maximal finite subgroups, is studied in [BH11].

1.8 Further reading

Elliptic genera: One of the original motivations behind tmf was Ochanine's definition of a genus of spin manifolds which takes values in the ring of modular forms for $\Gamma_0(2)$, which interpolates between the $\widehat{A}$-genus and the signature. Witten gave an interpretation of this genus in terms of 2-dimensional field theory, and produced a new genus (the *Witten genus*) of *string manifolds* valued in modular forms for $SL_2(\mathbb{Z})$ [Wit88], [Wit99]. These genera were refined to an orientation of elliptic spectra by Ando-Hopkins-Strickland [AHS01], and were shown to give E_∞ orientations

$$MString \to \mathrm{tmf},$$
$$MSpin \to \mathrm{tmf}_0(2)$$

by Ando-Hopkins-Rezk [AHR10] and Wilson [Wil15], respectively.

Geometric models: The most significant outstanding problem in the theory of topological modular forms is to give a geometric interpretation of this cohomology theory (analogous to the fact that K-theory classes are represented by vector bundles). Motivated by the work of Witten described above, Segal proposed that 2-dimensional field theories should represent tmf-cocycles [Seg07]. This idea has been fleshed out in detail by Stolz and Teichner, and concrete conjectures are proposed in [ST11]. While these conjectures have proved difficult to verify, partial results have been made by many researchers (see, for example, [BET18]).

Computations of the homotopy groups of $\mathrm{TMF}_0(N)$: Mahowald and Rezk computed the descent spectral sequence for $\pi_*\mathrm{TMF}_0(3)$ in [MR09], and a similar computation of $\pi_*\mathrm{TMF}_0(5)$ was performed by Ormsby and the author in [BO16] (see also [HHR17]). Meier gave a general additive description of $\pi_*\mathrm{TMF}_0(N)^\wedge_2$ for all N with $4 \nmid \varphi(N)$ in [Mei18].

Self-duality: Stojanoska showed that Serre duality for the stack $\overline{\mathcal{M}}_{ell}$ lifts to a self-duality result for Tmf[1/2]. This result was extended to $\mathrm{Tmf}_1(N)$ by Meier [Mei18]. Other self-duality results can be found in [MR99], [Beh06], [Bob18], and [BR]. Closely related to duality is the study of the Picard group of modules over various tmf spectra - see [MS16], [BBHS19].

Detection of the divided β-family: Adams used K-theory to define his e-invariant, and deduced that the order of the image of the J-homomorphism in degree $2k-1$ is given by the denominator of the Bernoulli number $\frac{B_k}{2k}$. The divided β-family, a higher chromatic generalization of the image of J, was constructed on the 2-line of the Adams-Novikov spectral sequence by Miller-Ravenel-Wilson [MRW77]. Laures used tmf to construct a generalization of the e-invariant, called the f-invariant [Lau99]. This invariant relates the divided beta family to certain congruences between modular forms [Beh09], [BL09], [vB16].

Quasi-isogeny spectra: The author showed that the Goerss-Henn-Mahowald-Rezk resolution of the 3-primary $K(2)$-local sphere [GHMR05] can be given a modular interpretation in terms of isogenies of elliptic curves [Beh06], and conjectured that something similar happens at all primes [Beh07].

The tmf resolution: Generalizing his seminal work on "bo-resolutions," Mahowald initiated the study of the tmf-based Adams spectral sequence. This was used in [BHHM08] to lift the 192-periodicity in $\mathrm{tmf}_{(2)}$ to a periodicity in the 2-primary stable homotopy groups of spheres, and in [BHHM17] to show coker J is non-trivial in "most" dimensions less than 140. The study of the tmf-based Adams spectral sequence begins with an analysis of the ring of cooperations $\mathrm{tmf}_*\mathrm{tmf}$. With 6 inverted, this was studied by Baker and Laures [Bak95], [Lau99]. The 2-primary structure of $\mathrm{tmf}_*\mathrm{tmf}$ was studied in [BOSS18].

Dyer-Lashof operations: Ando observed that power operations for elliptic cohomology are closely related to isogenies of elliptic curves [And00]. Following this thread, Rezk used the geometry of elliptic curves to compute the Dyer-Lashof algebra for the Morava E-theory E_2 at the prime 2 [Rez08]. This was generalized by Zhu to all primes [Zhu18b]. Using Rezk's "modular isogeny complex" [Rez12], Zhu was able to derive information about unstable homotopy groups of spheres [Zhu18a].

Spectral algebraic geometry: As mentioned in the introduction, Lurie introduced the notion of *spectral algebraic geometry*, and used it to give a revolutionary new construction of tmf [Lur09] (see also [Lur18a] and [Lur18b]).

Equivariant TMF: Grojnowski introduced the idea of complex analytic *equivariant elliptic cohomology* [Gro07] (see also [Rez16]). A variant based on K-theory was introduced by Devoto [Dev96] (see also [Gan13], [Kit14], [Hua18]). This idea was refined in the rational setting by Greenlees [Gre05]. Lurie used his spectral algebro-geometric construction of TMF to construct equivariant TMF (this is outlined in [Lur09], see [Lur18a] and [Lur18b] for more details). Equivariant elliptic cohomology was used by Rosu in his proof of the rigidity of the elliptic genus [Ros01].

K3 cohomology: Morava and Hopkins suggested that cohomology theories should be also be associated to K3 surfaces. Szymik showed this can be done in [Szy10] (see also [Pro18]).

Computations of π_*TAF: Very little is known about the homotopy groups of spectra of topological automorphic forms, for the simple reason that, unlike the modular case, very few computations of rings of classical integral automorphic forms exist in the literature. Nevertheless, special instances have been computed in [BL10], [HL10], [BL11], [LN12], [Law15], [vBT16], [vBT17].

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