β-FAMILY CONGRUENCES AND THE f-INVARIANT

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Abstract. In previous work, the authors have each introduced methods for studying the 2-line of the p-local Adams-Novikov spectral sequence in terms of the arithmetic of modular forms. We give the precise relationship between the congruences of modular forms introduced by the first author with the Q-spectrum and the f-invariant of the second author. This relationship enables us to refine the target group of the f-invariant in a way which makes it more manageable for computations.

1. Introduction

In [Ada66], J.F. Adams studied the image of the J-homomorphism

$$J : \pi_t(SO) \to \pi_t^S$$

by introducing a pair of invariants

$$d = d_t : \pi_t^S \to \pi_t K,$$
$$e = e_t : \ker(d_t) \to \mathrm{Ext}_{\mathcal{A}}^{1,t+1}(K_*, K_*)$$

where $\mathcal{A}$ is a certain abelian category of graded abelian groups with Adams operations. (Adams also studied analogs of d and e using real K-theory, to more fully detect 2-primary phenomena.) In order to facilitate the study of the e-invariant, Adams used the Chern character to provide a monomorphism

$$\theta_S : \mathrm{Ext}_{\mathcal{A}}^{1,t+1}(K_*, K_*) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

Thus, the e-invariant may be regarded as taking values in $\mathbb{Q}/\mathbb{Z}$. Furthermore, he showed that for t odd, and $k = (t+1)/2$, the image of θ_S is the cyclic group of order $\mathrm{denom}(B_k/2k)$, where B_k is the kth Bernoulli number.

The d and e-invariants detect the 0 and 1-lines of the Adams-Novikov spectral sequence (ANSS). In [Lau99], the second author studied an invariant

$$f : \ker(e_t) \to \mathrm{Ext}^{2,t+2}_{\mathrm{TMF}_* \mathrm{TMF}[\frac{1}{6}]}(\mathrm{TMF}[\tfrac{1}{6}]_*, \mathrm{TMF}[\tfrac{1}{6}]_*)$$

which detects the 2-line of the ANSS for π_*^S away from the primes 2 and 3. He furthermore used H. Miller's elliptic character to show that, if t is even and $k = (t+2)/2$, there is a monomorphism

$$\iota^2 : \mathrm{Ext}^{2,t+2}_{\mathrm{TMF}_* \mathrm{TMF}[\frac{1}{6}]}(\mathrm{TMF}[\tfrac{1}{6}]_*, \mathrm{TMF}[\tfrac{1}{6}]_*) \hookrightarrow D_{\mathbb{Q}}/(D_{\mathbb{Z}[\frac{1}{6}]} + (M_0)_{\mathbb{Q}} + (M_k)_{\mathbb{Q}}),$$

where D is Katz's ring of divided congruences and M_k is the space of weight k modular forms of level 1 meromorphic at the cusp. It is natural to ask for a description of the image of the map ι^2 in arithmetic terms.

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Remark 1.1. In [Lau99], the second author works with more general congruence subgroups $\Gamma \subseteq SL_2(\mathbb{Z})$ and associated cohomology theories E^Γ which also lead to results for the primes 2 and 3. The spectrum TMF is just the spectrum $E^{SL_2(\mathbb{Z})}$ when 6 is inverted. In this paper we shall not be considering the f invariant associated to more general congruence subgroups Γ and 6 shall always be a unit.

Attempting to generalize the J fiber-sequence

$$J \to KO_p \xrightarrow{\psi^\ell - 1} KO_p$$

the first author introduced a ring spectrum $Q(\ell)$ built from a length two TMF_p-resolution. In [Beh09, Thm. 12.1], it was shown that for $p \geq 5$, the elements $\beta_{i/j,k} \in (\pi_*^S)_p$ of [MRW77] are detected in the Hurewicz image of $Q(\ell)$. This gives rise to the association of a modular form $f_{i/j,k}$ to each element $\beta_{i/j,k}$. Furthermore, the forms $f_{i/j,k}$ are characterized by certain arithmetic conditions.

The purpose of this paper is to prove that the f-invariant of $\beta_{i/j,k}$ is given by the formula

$$f(\beta_{i/j,k}) = \frac{f_{i/j,k}}{p^k E_{p-1}^j} \qquad \text{(Theorem 4.2)}.$$

In particular, since the 2-line of the ANSS is generated by the elements $\beta_{i/j,k}$, the p-component of the image of the map ι^2 is characterized by the arithmetic conditions satisfied by the elements $f_{i/j,k}$.

J. Hornbostel and N. Naumann [HN07] computed the f invariant of the elements $\beta_{i/1,1}$ in terms of Katz's Artin-Schreier generators of the ring of p-adic modular forms. While their result is best suited to describe f-invariants of infinite families, it is difficult to explicitly get one's hands on their output. Direct computations with q-expansions are limited by the computability of q-expansions of modular forms, hence are generally not well suited for infinite families of computations. In low degrees, however, our formula can directly be used to compute with q-expansions. We demonstrate this by giving some sample calculations of some f-invariants at the prime 5.

Remark 1.2. It is natural to ask if the results of this paper can be extended to the primes 2 and 3. A difficulty arises because the cohomology theory TMF fails to be Landweber exact without inverting 6, and this in turn is related to the fact that the associated moduli stack of elliptic curves has geometric points with automorphism groups divisible by the primes 2 and 3. If one substitutes the group $SL_2(\mathbb{Z})$ with a small enough congruence subgroup so that the associated moduli stack is actually an algebraic space, then the corresponding f-invariant detects the 2-line of the 2 and 3-primary Adams-Novikov spectral sequences. However, the results of [Beh09] break down, because they rely on the approximation theorem of [BL06], and the analog of this approximation theorem for these congruence subgroups does not hold. In fact, the approximation theorem is not even true at the prime 2 for the full congruence subgroup $SL_2(\mathbb{Z})$.

We outline the organization of this paper. In Section 2, we review the f-invariant. In Section 3, we review the spectrum $Q(\ell)$, and use it to construct an invariant f' so that

$$f_{i/j,k} = f'(\beta_{i/j,k}).$$

In Section 4 we show that the f-invariant is directly expressible in terms of the invariant f'. In Section 5, we give our sample 5-primary calculations.

2. The f-invariant

This section reviews the f-invariant and its various aspects in homotopy theory and geometry. Our main sources are [Lau00] and [Lau99].

Theorem 2.1. *Let D be the ring of divided congruences defined by N. Katz in* [Kat73]*, that is, the ring of all inhomogeneous modular forms for $SL_2(\mathbb{Z})$ whose q-expansion is integral, and let M_t be the subspace of modular forms of homogeneous weight t. Then for all $k > 0$ there is a homomorphism*

$$f : \pi_{2k}^S \longrightarrow D_{\mathbb{Q}}/(D_{\mathbb{Z}[\frac{1}{6}]} \oplus (M_0)_{\mathbb{Q}} \oplus (M_{k+1})_{\mathbb{Q}})$$

whose kernel is the 3rd Adams-Novikov filtration for $MU[\frac{1}{6}]$.

Remark 2.2. In [Lau99], the second author actually defines the f invariant to take values in the subspace of

$$D_{\mathbb{Q}}/(D_{\mathbb{Z}[\frac{1}{6}]} \oplus (M_0)_{\mathbb{Q}} \oplus (M_{k+1})_{\mathbb{Q}})$$

spanned by inhomogeneous sums of modular forms of weights between 0 and $k+1$. Of course, there is no harm in regarding the invariant as taking values in the larger group above.

The construction of f is closely related to the construction of the classical e-invariant by F. Adams (see [Ada66]). Let T be a flat ring spectrum and let

$$s : X \longrightarrow Y$$

be a stable map from a finite spectrum into an arbitrary one. Suppose further that the d-invariant of s vanishes. This simply means that s vanishes in T homology. Then we have a short exact sequence

$$T_*Y \longrightarrow T_*C_s \longrightarrow T_*\Sigma X,$$

where C_s is the cofiber of s. We can think of the sequence as an extension of T_*X by T_*Y as a T_*T-comodule. This is the classical e-invariant of s in T-theory.

Next, suppose that

$$e(s) \in \mathrm{Ext}_{T_*T}(T_*X, T_*Y)$$

vanishes, that is, the exact sequence of T_*T-comodules splits and we choose a splitting. We also choose a T-monomorphism

$$\iota : Y \longrightarrow I$$

into a T-injective spectrum I. For instance, we can take $I = T \wedge Y$. Then there is a map

$$t : C_s \longrightarrow I$$

which is the image of ι_* under the induced splitting map

$$[Y, I] \cong \mathrm{Hom}_{T_*T}(T_*Y, T_*I) \longrightarrow \mathrm{Hom}_{T_*T}(T_*C_s, T_*I) \cong [C_s, I].$$

In particular, the map t coincides with ι on Y. Let F be the fiber of the map ι. Then s lifts to a map

$$\bar{s} : X \longrightarrow F$$

which makes the diagram

$$\begin{tikzcd} \Sigma^{-1}C_s \arrow[r] \arrow[d, "\Sigma^{-1}t"] & X \arrow[r, "s"] \arrow[d, "\bar{s}"] & Y \arrow[d, "id"] \\ \Sigma^{-1}I \arrow[r] & F \arrow[r] & Y \end{tikzcd}$$

commute.

Lemma 2.3. $d(\bar{s}) = 0$.

Proof. In the split exact sequence

$$\mathrm{Hom}_{T_*T}(T_*\Sigma X, T_*\Sigma F) \longrightarrow \mathrm{Hom}_{T_*T}(T_*C_s, T_*\Sigma F) \longrightarrow \mathrm{Hom}_{T_*T}(T_*Y, T_*\Sigma F)$$

the map $\Sigma\bar{s}_*$ restricted to C_s is in the image of the splitting and hence has to vanish. The claim follows since the map from C_s to ΣX is surjective in T-homology. □

Lemma 2.3 implies that we again get a short exact sequence

$$T_*F \longrightarrow T_*C_{\bar{s}} \longrightarrow T_*\Sigma X$$

which we can splice together with the short exact sequence

$$T_*\Sigma^{-1}Y \longrightarrow T_*\Sigma^{-1}I \longrightarrow T_*F.$$

This gives an extension of $T_*\Sigma^{-1}Y$ by $T_*\Sigma X$ of length 2, that is, an element

$$f(s) \in \mathrm{Ext}^2_{T_*T}(T_*X, T_*Y).$$

In the case $X = S^{2k}$, $Y = S^0$ and $T = \mathrm{TMF}[\frac{1}{6}]$, the image of $f(s)$ under the injection

$$\iota^2 : \mathrm{Ext}^2 \hookrightarrow D_{\mathbb{Q}}/(D_{\mathbb{Z}[\frac{1}{6}]} \oplus (M_0)_{\mathbb{Q}} \oplus (M_{k+1})_{\mathbb{Q}})$$

is the second author's f-invariant. The map ι^2 will be reviewed in Section 4.

We close this section with an alternative description of the f-invariant. First recall from [Lau00] that a framed manifold M represents a framed bordism class in second Adams-Novikov filtration if and only if it is the corner of a $(U, fr)^2$ manifold W. The boundary of W is decomposed into two manifolds with boundaries W^0 and W^1. The stable tangent bundle of W comes with a splitting

$$TW \cong (TW)^0 \oplus (TW)^1$$

and the bundles $(TW)^i$ are trivialized on W^i. Therefore, we get associated classes

$$(TW)^i \in K(W, W^i).$$

Let $\exp_T$ be the usual parameter for the universal Weierstrass cubic

$$y^2 = 4x^3 - E_4x + E_6$$

and let

$$\exp_K(x) = 1 - e^{-x}$$

be the standard parameter for the multiplicative formal group. Then following theorem is a consequence of Proposition 4.1.4 of [Lau00] after applying the complex orientation of the $\langle 2\rangle$-spectrum

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & K \arrow[d] \\ T \arrow[r] & K \wedge T \end{tikzcd} .$$

Theorem 2.4. *Let s be represented by M under the Pontryagin-Thom isomorphism. Then we have*

$$f(s) = \left\langle \prod_{i,j} \frac{x_i y_j}{\exp_K(x_i)\exp_T(y_j)}, [W, \partial W] \right\rangle.$$

Here, (x_i) and (y_j) are the formal Chern roots of $(TW)^0$ and $(TW)^1$ respectively.

We remark that there also are descriptions of the f-invariant in terms of a spectral invariant which is analogous to the classical relation between the e-invariant and the η-invariant. We refer the reader to [vB08] and [BN].

3. The spectrum $Q(\ell)$ and the invariant f'

For a $\mathbb{Z}[1/N]$-algebra R we shall let $M_k(\Gamma_0(N))_R$ denote the space of modular forms of weight k over R of level $\Gamma_0(N)$ which are meromorphic at the cusps. For $N = 1$ we shall simplify the notation by writing

$$(M_k)_R := M_k(\Gamma_0(1))_R.$$

Let $\mathrm{TMF}_0(N)$ denote the corresponding spectrum of topological modular forms with N inverted (see [Beh06, Sec. 1.2.1], [Beh07, Sec. 5]). For primes $p > 3$, $\pi_* \mathrm{TMF}_0(N)_p$ is concentrated in even degrees, and we have

$$\pi_{2k} \mathrm{TMF}_0(N)_p \cong M_k(\Gamma_0(N))_{\mathbb{Z}_p}. \tag{3.1}$$

Remark 3.2. One could view the isomorphism of (3.1) as a consequence of the fact that the spectrum $\mathrm{TMF}_0(N)[\frac{1}{6}]$ is equivalent to the spectrum $E^{\Gamma_0(N)}$ of [Lau99], or as a consequence of the fact that the descent spectral sequence

$$H^s(\mathcal{M}_{ell}^{\Gamma_0(N)}[\tfrac{1}{6}], \omega^{\otimes t}) \Rightarrow \pi_{2t-s} \mathrm{TMF}_0(N)[\tfrac{1}{6}]$$

is concentrated on $s = 0$.

Fix a pair of distinct primes p and ℓ. In [Beh06], the first author introduced a p-local spectrum $Q(\ell)$, defined as the totalization of a certain semi-cosimplicial spectrum

$$Q(\ell) = \mathrm{Tot}(Q(\ell)^\bullet)$$

where $Q(\ell)^\bullet$ has the form

$$Q(\ell)^\bullet = \left(\mathrm{TMF}_p \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \mathrm{TMF}_0(\ell)_p \\ \times \\ \mathrm{TMF}_p \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \mathrm{TMF}_0(\ell)_p \right). \tag{3.3}$$

In [Beh09, Sec. 4] the spectrum $Q(\ell)$ is reinterpreted as the smooth hypercohomology of a certain open subgroup of an adele group acting on a certain spectrum. The semi-cosimplicial spectrum $Q(\ell)^\bullet$ is actually a semi-cosimplicial E_∞-ring spectrum, so the spectrum $Q(\ell)$ is an E_∞-ring spectrum. In particular, there is a unit map

$$\eta : S \to Q(\ell). \tag{3.4}$$

The spectrum $Q(\ell)$ is designed to be an approximation of the $K(2)$-local sphere. More precisely, the spectrum $Q(\ell)_{K(2)}$ is given as the homotopy fixed points of a subgroup

$$\Gamma_\ell \subset \mathbb{S}_2 \tag{3.5}$$

of the Morava stabilizer group acting on the Morava E-theory E_2 [Beh07] and this subgroup is dense if ℓ generates $\mathbb{Z}_p^\times$ [BL06]. The spectrum $Q(\ell)$ is $E(2)$-local. In [Beh09, Thm. 12.1] it is proven that elements $\beta_{i/j,k} \in \pi_*(S_{E(2)})$ of [MRW77] are detected by the map

$$S_{E(2)} \to Q(\ell).$$

(It is not known if $Q(\ell)$ detects the entire divided beta family at the primes 2 and 3.)

Taking the homotopy groups of the semi-cosimplicial spectrum $Q(\ell)^\bullet$ (3.3) gives a semi-cosimplicial abelian group

$$C(\ell)^\bullet_{2k} := \left((M_k)_{\mathbb{Z}_p} \begin{array}{c} \to \\ \to \end{array} \begin{array}{c} M_k(\Gamma_0(\ell))_{\mathbb{Z}_p} \\ \times \\ (M_k)_{\mathbb{Z}_p} \end{array} \begin{array}{c} \to \\ \to \\ \to \end{array} M_k(\Gamma_0(\ell))_{\mathbb{Z}_p} \right). \tag{3.6}$$

It is shown in [Beh09, Sec. 6] that the morphisms

$$d_0, d_1 : (M_k)_{\mathbb{Z}_p} \to M_k(\Gamma_0(\ell))_{\mathbb{Z}_p} \times (M_k)_{\mathbb{Z}_p},$$

induced by the initial coface maps of the cosimplicial abelian group $C(\ell)^\bullet_{2k}$, are given on the level of q-expansions by

$$d_0(f(q)) := (\ell^k f(q^\ell), \ell^k f(q)), \tag{3.7}$$

$$d_1(f(q)) := (f(q), f(q)). \tag{3.8}$$

The Bousfield-Kan spectral sequence for computing $\pi_* \operatorname{Tot}(Q(\ell)^\bullet)$ gives a spectral sequence

$$H^s(C(\ell)^\bullet)_t \Rightarrow \pi_{t-s} Q(\ell). \tag{3.9}$$

For $p > 3$, this spectral sequence collapses for dimensional reasons [Beh09, Cor. 5.2], giving us the following lemma.

Lemma 3.10. *The edge homomorphism*

$$H^2(C(\ell)^\bullet)_t \to \pi_{t-2}(Q(\ell))$$

is an isomorphism for $t \equiv 0 \mod 4$.

Lemma 3.11. *There is a map of spectral sequences*

$$\begin{tikzcd} \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s} S_{(p)} \arrow[d, "\eta_*"] \\ H^s(C(\ell)^\bullet)_t \arrow[r, Rightarrow] & \pi_{t-s} Q(\ell) \end{tikzcd}$$

from the Adams-Novikov spectral sequence for the sphere to the Bousfield-Kan spectral sequence for $Q(\ell)$.

To prove Lemma 3.11 we shall need the following lemma.

Lemma 3.12. *Suppose that $R^\bullet$ is a semi-cosimplicial commutative S-algebra, E is a commutative S-algebra, and $\phi : E \to R^0$ is a map of commutative S-algebras. Then there is a canonical extension of ϕ to a map of semi-cosimplicial commutative S-algebras*

$$\phi^\bullet : E^{\wedge \bullet + 1} \to R^\bullet$$

where

$$E^{\wedge \bullet+1} = \left(E \quad \underset{1\wedge\eta}{\overset{\eta\wedge 1}{\rightrightarrows}} \quad E\wedge E \quad \begin{array}{c} \xrightarrow{\eta\wedge 1\wedge 1} \\ \xrightarrow{1\wedge\eta\wedge 1} \\ \xrightarrow{1\wedge 1\wedge \eta} \end{array} \quad E\wedge E\wedge E \qquad \cdots \right)$$

is the canonical cosimplicial E-resolution of the sphere.

Proof. A semi-cosimplicial commutative S-algebra is a functor

$$\Delta_{inj} \to \{\text{commutative } S\text{-algebras}\},$$

where Δ_{inj} is the category of finite ordered sets and order preserving injections. Let $\underline{m}$ be the object of Δ_{inj} given by

$$\underline{m} = \{0, 1, \dots, m\}$$

and for $0 \le i \le n$ define $\iota_i^m : \underline{0} \to \underline{m}$ by $\iota_i^m(0) = i$. The map ϕ^s is defined to be the composite

$$E^{\wedge s+1} \xrightarrow{((\iota_0^s)_* \circ \phi)\wedge \cdots \wedge ((\iota_n^n)_* \circ \phi)} (R^s)^{\wedge s+1} \xrightarrow{\mu_{s+1}} R^s$$

where μ_{s+1} denotes the $s+1$-fold product. The maps ϕ^s are easily seen to assemble into a map of semi-cosimplicial spectra. ☐

Proof of Lemma 3.11. Lemma 3.12 implies that there exists a map of semi-cosimplicial spectra

$$1^\bullet : \mathrm{TMF}_p^{\wedge \bullet+1} \to Q(\ell)^\bullet$$

and hence a map from the Bousfield-Kan spectral sequence for $\mathrm{TMF}_p^{\wedge\bullet+1}$ to the Bousfield-Kan spectral sequence for $Q(\ell)^\bullet$. However, since $\mathrm{TMF}_p^{\bullet+1}$ is the canonical TMF_p-injective resolution of S, the Bousfield-Kan spectral sequence for $\mathrm{TMF}_p^{\wedge\bullet+1}$ is the TMF_p-Adams-Novikov spectral sequence for S. Since TMF_p is complex orientable, there is a map of ring spectra $BP \to \mathrm{TMF}_p$, and hence a map from the BP-Adams-Novikov spectral sequence to the TMF_p-Adams-Novikov spectral sequence. ☐

The short exact sequences of BP_*BP-comodules

$$0 \to BP_* \to BP_*[p^{-1}] \to BP_*/p^\infty \to 0,$$
$$0 \to BP_*/p^\infty \to BP_*/p^\infty[v_1^{-1}] \to BP_*/(p^\infty, v_1^\infty) \to 0$$

give rise to long exact sequences in Ext, and the connecting homomorphisms give a composite

$$\begin{aligned} \delta_{v_1,p} : \mathrm{Ext}^{0,t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)) &\xrightarrow{\delta_{v_1}} \mathrm{Ext}^{1,t}_{BP_*BP}(BP_*, BP_*/p^\infty) \\ &\xrightarrow{\delta_p} \mathrm{Ext}^{2,t}_{BP_*BP}(BP_*, BP_*). \end{aligned} \tag{3.13}$$

The computations of [MRW77] imply the following lemma.

Lemma 3.14. *The homomorphism $\delta_{v_1,p}$ of (3.13) is an isomorphism for $t > 0$.*

Since the spectrum $\mathrm{TMF}[\frac{1}{6}]$ is Landweber exact, the spectrum TMF_p is complex orientable. Since TMF_p is p-local, it admits a p-typical complex orientation, and a choice of p-typical complex orientation

$$BP \to \mathrm{TMF}_p \to \mathrm{TMF}_0(\ell)_p$$

sends v_1 to a non-zero multiple of the Hasse invariant E_{p-1} mod p. The complex $C(\ell)^\bullet/p^k$ is a complex of modules over the ring $\mathbb{Z}_p[v_1^{p^{k-1}}]$. The short exact sequences

$$0 \to C(\ell)^\bullet \to C(\ell)^\bullet[p^{-1}] \to C(\ell)^\bullet/p^\infty \to 0,$$
$$0 \to C(\ell)^\bullet/p^\infty \to C(\ell)^\bullet/p^\infty[v_1^{-1}] \to C(\ell)^\bullet/(p^\infty, v_1^\infty) \to 0$$

give rise to long exact sequences in H^*, and the connecting homomorphisms give a composite

$$\delta_{v_1,p} : H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_t \xrightarrow{\delta_{v_1}} H^1(C(\ell)^\bullet/p^\infty)_t \xrightarrow{\delta_p} H^2(C(\ell)^\bullet)_t.$$

Using Lemmas 3.10 and 3.14, we have the following diagram, for $t > 0$.

(3.15)
$$\begin{tikzcd}
\pi_{4t-2}S_{(p)} \arrow[rr] & & \pi_{4t-2}Q(\ell) \\
\mathrm{Ext}^{2,4t}_{BP_*BP_*}(BP_*, BP_*) \arrow[u, Rightarrow] \arrow[rr] \arrow[drr, dotted, "f'"] & & H^2(C(\ell)^\bullet)_{4t} \arrow[u, "\cong"'] \\
\mathrm{Ext}^{0,4t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)) \arrow[u, "\delta_{v_1,p}", "\cong"'] \arrow[rr] & & H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t} \arrow[u, "\delta_{v_1,p}"']
\end{tikzcd}$$

Since p is odd and $\mathrm{Ext}^{2,m}_{BP_*BP}(BP_*, BP_*)$ is concentrated in degrees $m \equiv 0 \mod 4$, the invariant f' may be regarded as an invariant defined on the entire 2-line of the ANSS. Moreover, because $\pi_{4t-2}S_{(p)}$ contains no elements of Adams-Novikov filtration less than 2, the invariant f' may be regarded as giving a homotopy invariant through the composite

$$\pi_{4t-2}S_{(p)} \to \mathrm{Ext}^{2,4t}_{BP_*BP}(BP_*, BP_*) \xrightarrow{f'} H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t}.$$

We shall find that this invariant f' is closely related to the f invariant of the second author.

We end this section by describing some of the salient features of the invariant f'. Namely, we shall show:

(i) the homomorphism f' is a monomorphism, and if ℓ generates $\mathbb{Z}_p^\times$, the homomorphism f' is almost an isomorphism, and
(ii) the groups $H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t}$ admit a precise arithmetic interpretation in terms of congruences of q-expansions of modular forms.

The injectivity and almost surjectivity of f'.

Because v_2 is invertible in $C(\ell)^\bullet/(p^\infty, v_1^\infty)$, there is a factorization

(3.16)
$$\begin{tikzcd}
\mathrm{Ext}^{2,4t}_{BP_*BP}(BP_*, BP_*) \arrow[r, "f'"] & H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t} \\
\mathrm{Ext}^{0,4t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)) \arrow[u, "\delta_{v_1,p}", "\cong"'] \arrow[r, "L_{v_2}"'] & \mathrm{Ext}^{0,4t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)[v_2^{-1}]) \arrow[u, dotted, "\bar{\eta}"']
\end{tikzcd}$$

Recall from [MRW77] that for $t > 0$ the groups

$$\mathrm{Ext}^{0,4t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)) \quad \text{and} \quad \mathrm{Ext}^{0,4t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)[v_2^{-1}])$$

are generated by elements $\beta_{i/j,k}$ for certain combinations of indices i, j, and k. As usual, $\beta_{i/j}$ denotes the element $\beta_{i/j,1}$.

Proposition 3.17.

(i) *The map* L_{v_2} *of (3.16) is injective, and the cokernel is an* $\mathbb{F}_p$*-vector space with basis*
$$\{\beta_{p^n/j} \,:\, n \geq 2, p^n < j \leq p^n + p^{n-1} - 1\}.$$
(ii) *The map* $\bar{\eta}$ *of (3.16) is injective, and if* ℓ *generates* $\mathbb{Z}_p^\times$*, it is an isomorphism.*

Proof. (i) follows directly from the calculations of [MRW77]. (ii) follows from the fact that the map $\bar{\eta}$ factors as

$$\begin{tikzcd}
\mathrm{Ext}^{0,4t}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)[v_2^{-1}]) \arrow[rr, "\bar{\eta}"] \arrow[dr, "\bar{\eta}'", "\cong"'] & & H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t} \\
& H^0_c(\mathbb{S}_2, \pi_*E_2/(p^\infty, v_1^\infty))^{\mathrm{Gal}}_{4t} \arrow[ur, "\bar{\eta}''"] &
\end{tikzcd}$$

where $\bar{\eta}'$ is the Morava change-of-rings isomorphism, and $\bar{\eta}''$ is given by the composite

$$H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t} \xrightarrow[\cong]{\omega} H^0(\Gamma_\ell, \pi_*M_2E_2)^{\mathrm{Gal}}_{4t} \xrightarrow{\nu} H^0_c(\mathbb{S}_2, \pi_*M_2E_2)^{\mathrm{Gal}}_{4t}.$$

Here, ω is the isomorphism given by [Beh09, Cor. 7.7] where Γ_ℓ is the subgroup of $\mathbb{S}_2$ of (3.5), the spectrum M_2E_2 is the second monochromatic layer of E_2, and ν is the monomorphism induced by the inclusion of the subgroup. Lemma 11.1 of [Beh09] states that ν is an isomorphism if ℓ generates $\mathbb{Z}_p^\times$ [Beh09, Lem. 11.1]. Note that the same argument in [Rav84, Thm. 6.1] computing π_*M_nBP applies to compute
$$\pi_*M_2E_2 \cong (\pi_*E_2)/(p^\infty, v_1^\infty).$$
□

We conclude that f' is injective, and if ℓ generates $\mathbb{Z}_p^\times$, the only generators of $H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))$ not in the image of f' are those corresponding to the Greek letter elements $\beta_{p^n/j}$ for $j > p^n$.

The arithmetic interpretation of the groups $H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))$. The groups $H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t}$ are computed by the colimit of groups
$$H^0(C(\ell)^\bullet/(p^\infty, v_1^\infty))_{4t} = \operatorname*{colim}_k \operatorname*{colim}_{\substack{j=sp^{k-1} \\ s\geq 1}} \mathcal{B}_{2t/j,k}$$
where
$$\mathcal{B}_{2t/j,k} = H^0(C(\ell)^\bullet/(p^k, v_1^j))_{4t+2j(p-1)}.$$
Using the fact that v_1 corresponds, modulo p, to a non-zero multiple of the Hasse invariant E_{p-1} in the ring of modular forms, we have
$$\mathcal{B}_{2t/j,k} = \ker\left(\frac{M_{2t+j(p-1)}}{(p^k, E_{p-1}^j)} \xrightarrow{d_0-d_1} \begin{array}{c} \frac{M_{2t+j(p-1)}}{(p^k, E_{p-1}^j)} \\ \oplus \\ \frac{M_{2t+j(p-1)}(\Gamma_0(\ell))}{(p^k, E_{p-1}^j)} \end{array} \right).$$

Serre [Kat73, Prop. 4.4.2] showed that two modular forms f_1 and f_2 over $\mathbb{Z}/p^k$ are linked by multiplication by E_{p-1}^j (for $j \equiv 0 \mod p^{k-1}$) if and only if the corresponding q-expansions satisfy

$$f_1(q) \equiv f_2(q) \mod p^k.$$

Using this, and (3.7)-(3.8), the following theorem is proven in [Beh09].

Theorem 3.18 ([Beh09, Thm. 11.3])**.** *There is a one-to-one correspondence between the additive generators of order p^k in $\mathcal{B}_{t/j,k}$ and the modular forms $f \in M_{t+j(p-1)}$ (modulo p^k) satisfying*

(1) *We have $t \equiv 0 \mod (p-1)p^{k-1}$.*
(2) *The q-expansion $f(q)$ is not congruent to 0 mod p.*
(3) *We have $\mathrm{ord}_q f(q) > \frac{t}{12}$ or $\mathrm{ord}_q f(q) = \frac{t-2}{12}$.*
(4) *There does not exist a form $f' \in M_{t'}$ such that $f'(q) \equiv f(q) \mod p^k$ for $t' < t + j(p-1)$.*
$(5)_\ell$ *There exists a form*

$$g \in M_t(\Gamma_0(\ell))$$

satisfying

$$f(q^\ell) - f(q) \equiv g(q) \mod p^k.$$

Remark 3.19. It follows from [Beh09, Cor. 11.7], that a modular form satisfying (1)–(5) corresponding to $f'(x)$ is independent of the choice of the prime ℓ.

4. The relation between f and f'

Let ℓ be a generator of $\mathbb{Z}_p^\times$. We start with a cohomology class

$$x \in \mathrm{Ext}^{2,2t}_{BP_*BP}(BP_*, BP_*)$$

with corresponding invariant

$$f'(x) \in \mathcal{B}_{t/j,k} = H^0(C^\bullet(\ell)/(p^k, v_1^j))_{2t+2j(p-1)}. \tag{4.1}$$

Note that since p is odd, t must be even. By Theorem 3.18, a representative of $f'(x)$ is a $\mathbb{Z}/p^k$ modular form φ of weight $t + j(p-1)$ for $SL_2(\mathbb{Z})$ which satisfies certain congruences. We view φ as a divided congruence, more precisely, as an element of

$$D \otimes \mathbb{Z}/p^k.$$

Theorem 4.2. *The f-invariant of the class x is given by*

$$p^{-k} E_{p-1}^{-j}(\varphi - q^0(\varphi))$$

where q^0 is the 0th Fourier coefficient, and j, k are given by (4.1).

The proof of Theorem 4.2 will be deferred to the end of the section.

Remark 4.3. For $t > 0$, Theorem 3.18(3) implies that there exists a representative φ of $f'(x)$ with $q^0(\varphi) = 0$. Since the modular form $f_{i/j,k}$ of [Beh09] is such a representative of $f'(\beta_{i/j,k})$, Theorem 4.2 implies that

$$f(\beta_{i/j,k}) = \frac{f_{i/j,k}}{p^k E_{p-1}^j}.$$

Corollary 4.4. *The class*
$$p^k E_{p-1}^j f(x)$$
is congruent to a $\mathbb{Z}/p^k$-modular form φ of weight $t+j(p-1)$ up to modular forms of weights $j(p-1)$ and $t+j(p-1)$. Moreover, φ satisfies the conditions (1)-(5) of 3.18.

Remark 4.5. We pause to explain how the expression in Theorem 4.2 may be regarded as an element of the subgroup
$$\frac{D_{\mathbb{Q}}}{D_{\mathbb{Z}_{(p)}} + (M_0)_{\mathbb{Q}} + (M_t)_{\mathbb{Q}}} \subset \frac{D_{\mathbb{Q}}}{D_{\mathbb{Z}[\frac{1}{6}]} + (M_0)_{\mathbb{Q}} + (M_t)_{\mathbb{Q}}}$$
in a way that more clearly accounts for the indeterminacy of the f-invariant. Katz showed that D is a dense subspace of $\mathbb{V}$, the ring of generalized p-adic modular functions [Kat75]. The ring $\mathbb{V}$ has an action by the group $\mathbb{Z}_p^\times$ through Diamond operators, and the weight t subspace $\mathbb{V}_t$ is canonically identified by
$$\mathbb{V}_t \cong (M_*)_{\mathbb{Z}_p}[E_{p-1}^{-1}]_t.$$
We therefore have
$$\frac{D_{\mathbb{Q}}}{D_{\mathbb{Z}_{(p)}} + (M_0)_{\mathbb{Q}} + (M_t)_{\mathbb{Q}}} \cong \frac{\mathbb{V}_{\mathbb{Q}}}{\mathbb{V} + (M_0)_{\mathbb{Q}_p} + (M_t)_{\mathbb{Q}_p}}.$$
Taking the weight t subspace we get
$$\begin{aligned} \frac{(\mathbb{V}_t)_{\mathbb{Q}}}{\mathbb{V}_t + (M_t)_{\mathbb{Q}_p}} &\cong \left(\frac{(M_*)_{\mathbb{Q}_p}[E_{p-1}^{-1}]}{(M_*)_{\mathbb{Z}_p}[E_{p-1}^{-1}] + (M_*)_{\mathbb{Q}_p}} \right)_t \\ &= \left(\frac{(M_*)_{\mathbb{Z}_p}}{(p^\infty, E_{p-1}^\infty)} \right)_t . \end{aligned}$$
The expression $p^{-k}E_{p-1}^{-j}\phi$ clearly may be regarded as an element of the group above.

Let T be $\mathrm{TMF}[\frac{1}{6}]$ and
$$M^{(2)} = \pi_* T \wedge T$$
be the Hopf algebroid of cooperations of T. An element of $M^{(2)}$ is a modular form in two variables which is meromorphic at ∞ and has (away from 6) an integral Fourier expansion (see [Lau99]).

Consider the map of semi-cosimplicial spectra
$$1^\bullet : \mathrm{TMF}_p^{\wedge \bullet+1} \to Q(\ell)^\bullet$$
of Lemma 3.12. Applying the functor $\pi_*(-)$, we get a map of semi-cosimplicial abelian groups
$$\pi_{2k}(T_p^{\bullet+1}) = M_k^{(\bullet+1)} \to C^\bullet(\ell)_{2k}$$
which in low degrees gives the following commutative diagram.
$$\begin{tikzcd} (M_k)_{\mathbb{Z}_p} \arrow[r, "d_0-d_1"] \arrow[d, "="] & (M_k^{(2)})_{\mathbb{Z}_p} \arrow[d, "\phi"] \\ (M_k)_{\mathbb{Z}_p} \arrow[r, "d_0-d_1"] & M_k(\Gamma_0(\ell))_{\mathbb{Z}_p} \times (M_k)_{\mathbb{Z}_p} \end{tikzcd} .$$

Lemma 4.6. *The induced map in cohomology*

$$H^0(M_*^{(\bullet+1)}/(p^\infty, E_{p-1}^\infty)) \longrightarrow H^0(C^\bullet(\ell)/(p^\infty, v_1^\infty))$$

is an isomorphism.

Proof. By [HS05], there is a change-of-rings isomorphism

$$\begin{aligned} H^0(M_*^{(\bullet+1)}/(p^\infty, E_{p-1}^\infty)) &= \mathrm{Ext}^0_{\mathrm{TMF}_* \mathrm{TMF}_p}(\pi_* \mathrm{TMF}_p, \pi_* \mathrm{TMF}_p /(p^\infty, E_{p-1}^\infty)) \\ &\cong \mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)[v_2^{-1}]). \end{aligned}$$

The lemma follows from the isomorphism $\bar{\eta}$ of Proposition 3.17. $\square$

Next we explain how to get from an element in

$$H^0(M_*/(p^\infty, E_{p-1}^\infty)) \cong \mathrm{Ext}^0_{M^{(2)}}(M_*, M_*/(p^\infty, E_{p-1}^\infty))$$

to a congruence in

$$D_{\mathbb{Q}}/(D_{\mathbb{Z}[\frac{1}{6}]} \oplus (M_0)_{\mathbb{Q}} \oplus (M_k)_{\mathbb{Q}}).$$

For this, we first describe how a class φ in

$$\mathrm{Ext}^0_{M^{(2)}}(M_*, M_*/(p^\infty, E_{p-1}^\infty))$$

gives rise to a class in

$$\mathrm{Ext}^2_{M^{(2)}}(M_*, M_*).$$

We use the geometric boundary theorem

Theorem 4.7. [Rav86] *Write $E_*(X)$ for the E_*-term of the T-based Adams Novikov spectral sequence which conditionally converges to the homotopy of the T-nilpotent completion of X. Let*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$$

be a cofiber sequence of finite spectra with $T_*(h) = 0$. Assume further that $[s] \in E_2^{t,*+t}(Y)$ converges to s. Then $\delta\,[s]$ converges to $h_*(s)$ where δ is the connecting homomorphism to the short exact sequence of chain complexes

$$0 \longrightarrow E_1(W) \longrightarrow E_1(X) \longrightarrow E_1(Y) \longrightarrow 0.$$

For a multi index I let

$$M(I) = M(i_0, \ldots, i_{n-1})$$

be the generalized Moore spectrum with

$$BP_*M(I) = \Sigma^{-||I||-n} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$$

where

$$||I|| = \sum_j 2i_j(p^j - 1)$$

Each $M(I)$ admits a self map

$$\Sigma^{2i_n(p^n-1)} M(I) \longrightarrow M(I)$$

which induces multiplication by $v_n^{i_n}$. Its fiber is $M(I, i_n)$. We apply the geometric boundary theorem to the sequences

$$\Sigma^{2i_1(p-1)} M(i_0) \xrightarrow{v_1^i} M(i_0) \longrightarrow \Sigma M(i_0, i_1) \longrightarrow \Sigma^{2i_1(p-1)+1} M(i_0)$$

and

$$S \xrightarrow{p^{i_0}} S \longrightarrow \Sigma M(i_0) \longrightarrow S^1 \ .$$

For

$$\varphi \in E_2^0(M(i_0, i_1)) = \mathrm{Ext}^0_{M^{(2)}}(M_*, M_*/(p^{i_0}, E_{p-1}^{i_1}))$$

we have

$$\delta\varphi = \left[\frac{d^0\varphi - d^1\varphi}{E_{p-1}^{i_1}}\right] \in E_2^1(M(i_0)) = \mathrm{Ext}^1_{M^{(2)}}(M_*, M_*/p^{i_0})$$

and

$$\delta\delta\varphi = \left[p^{-i_0}\sum_{i=0}^{2}(-1)^i d^i \left[\frac{d^0\varphi - d^1\varphi}{E_{p-1}^{i_1}}\right]\right] \in E_2^2(S) = \mathrm{Ext}^2_{M^{(2)}}(M_*, M_*)$$

where d^i denote the differentials of the cobar complex

$$(\Omega_T^\bullet)_{2k} = \pi_{2k}T^{\bullet+1} \cong M_k^{(\bullet+1)}.$$

The maps of ring spectra

$$T \xrightarrow{q^0} K_{\mathbb{Z}[1/6]} \xrightarrow{ch^0} H_{\mathbb{Q}}$$

induce the following map of semi-cosimplicial spectra.

$$\begin{tikzcd}
T \arrow[r, shift left, "\eta\wedge 1"] \arrow[r, shift right, "1\wedge\eta"'] \arrow[d, "1"] & T\wedge T \arrow[r, shift left=2, "\eta\wedge 1\wedge 1"] \arrow[r, "1\wedge\eta\wedge 1"] \arrow[r, shift right=2, "1\wedge 1\wedge\eta"'] \arrow[d, "q^0\wedge 1"] & T\wedge T\wedge T \arrow[d, "ch^0\circ q^0\wedge q^0\wedge 1"] \\
T \arrow[r, shift left, "\eta\wedge 1"] \arrow[r, shift right, "q^0\wedge\eta"'] & K_{\mathbb{Z}[1/6]}\wedge T \arrow[r, shift left=2, "\eta\wedge 1\wedge 1"] \arrow[r, "ch^0\wedge\eta\wedge 1"] \arrow[r, shift right=2, "ch^0\wedge q^0\wedge\eta"'] & H_{\mathbb{Q}}\wedge K_{\mathbb{Z}[1/6]}\wedge T
\end{tikzcd}$$

Taking $\pi_{2k}(-)$, and using [Lau99, Thm. 2.7], we get the following map of semi-cosimplicial abelian groups

$$\begin{tikzcd}
(\Omega_T^\bullet)_{2k} \arrow[d, "\rho"] & M_k^{(1)} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "="', "\rho^0"] & M_k^{(2)} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] \arrow[d, "\rho^1"] & M_k^{(3)} \arrow[d, "\rho^2"] \\
(\Omega^\bullet_{T,K,H})_{2k} & (M_k)_{\mathbb{Z}[\frac{1}{6}]} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & D_{\mathbb{Z}[\frac{1}{6}]} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & D_{\mathbb{Q}}
\end{tikzcd}$$

In $(\Omega^\bullet_{T,K,H})_{2k}$, we have

$$d_1(D_{\mathbb{Z}[\frac{1}{6}]}) \subseteq (M_k)_{\mathbb{Q}} \subseteq D_{\mathbb{Q}},$$
$$d_2(D_{\mathbb{Z}[\frac{1}{6}]}) \subseteq (M_0)_{\mathbb{Q}} \subseteq D_{\mathbb{Q}}.$$

Therefore, by modding out by these subgroups of $D_{\mathbb{Q}}$, we get a map:

$$\begin{tikzcd}
(\Omega_T^\bullet)_{2k} \arrow[d, "\bar\rho"] & M_k^{(1)} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "="', "\bar\rho^0"] & M_k^{(2)} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] \arrow[d, "\bar\rho^1"] & M_k^{(3)} \arrow[d, "\bar\rho^2"] \\
(\bar\Omega^\bullet_{T,K,H})_{2k} & (M_k)_{\mathbb{Z}[\frac{1}{6}]} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & D_{\mathbb{Z}[\frac{1}{6}]} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & D_{\mathbb{Q}}/((M_0)_{\mathbb{Q}} + (M_k)_{\mathbb{Q}})
\end{tikzcd}$$

The first coface maps of the semi-cosimplicial abelian group $(\bar{\Omega}^{\bullet}_{T,K,H})_{2k}$ are given by
$$d^0 = \iota,\ d^1 = q^0$$
and the second ones by
$$d^0 = \iota,\ d^1 = d^2 = 0$$
where ι the canonical inclusion. The induced map in cohomology is the inclusion
$$\iota^2 : \mathrm{Ext}^2_{M^{(2)}}(M_*, M_*) \hookrightarrow D_{\mathbb{Q}}/((D)_{\mathbb{Z}[1/6]} \oplus (M_0)_{\mathbb{Q}} \oplus (M_k)_{\mathbb{Q}}).$$
Hence we have
$$\bar{\rho}_* \delta\delta\varphi = p^{-i_0} E_{p-1}^{-i_1}(\varphi - q^0(\varphi))$$
and the proof of the theorem is completed.

5. Examples at $p = 5$

Below are some computations of the q-expansions of the modular forms $f_{i/j,k}$ representing $f'(\beta_{i/j,k})$ at $p = 5$. The q-expansions of the corresponding f invariants, by Theorem 4.2, are given by
$$f(\beta_{i/j,k}) = p^{-k} E_{p-1}^{-j} f_{i/j,k}(q).$$
The computations were performed using the MAGMA computer algebra system, with $\ell = 2$, as follows.

(i) A basis $\{F_\alpha(q)\}$ of q-expansions of forms in M_{24i} satisfying Theorem 3.18(3) was generated.
(ii) A basis $\{G_\beta(q)\}$ of q-expansions of holomorphic forms in $M_{24i-4j}(\Gamma_0(\ell))_{\mathbb{Z}/5^k}$ was generated.
(iii) Basic linear algebra is used to calculate a basis of linear combinations $\sum_\alpha a_\alpha F_\alpha$ such that
$$\sum_\alpha a_\alpha (F_\alpha(q^2) - F_\alpha(q)) \equiv \sum_\beta b_\beta G_\beta(q) \mod 5^k.$$

Note 5.1. The following modular forms are normalized so that the leading term has coefficient 1. Therefore, they may differ from the f'-invariants of $\beta_{i/j,k}$ by a multiple in $\mathbb{Z}_p^\times$.

$f_{1/1,1} = \quad \Delta^2 =$

```
q^2 + 2*q^3 + q^7 + q^12 + 2*q^13 + q^17 + 2*q^18 +
2*q^22 + 2*q^23 + 3*q^28 + q^32 + 4*q^33 + q^37 +
2*q^42 + 2*q^43 + q^47 + 2*q^48 + q^52 + 2*q^53 +
2*q^62 + 2*q^63 + q^67 + 3*q^68 + 2*q^73 + 2*q^77 +
4*q^78 + 2*q^82 + 2*q^83 + q^92 + 4*q^93 + q^97 +
3*q^98 + O(q^100) mod 5
```

$f_{2/1,1} = \quad \Delta^4 =$

```
q^4 + 4*q^5 + 4*q^6 + 2*q^9 + 4*q^10 + 3*q^14 +
3*q^15 + 3*q^16 + 4*q^19 + 2*q^20 + 3*q^21 + 2*q^24
+ 2*q^26 + q^29 + 3*q^30 + 2*q^34 + 4*q^35 + 3*q^36
+ 3*q^39 + 2*q^44 + 3*q^45 + q^51 + 4*q^54 + 3*q^55
+ q^56 + 2*q^59 + 4*q^60 + 2*q^64 + 3*q^65 + 3*q^66
+ 4*q^69 + 4*q^70 + 2*q^76 + q^79 + 4*q^80 + 4*q^81
+ q^84 + 4*q^85 + q^86 + 3*q^89 + 3*q^90 + q^91 +
4*q^94 + 4*q^96 + 4*q^99 + O(q^100) mod 5
```

$f_{3/1,1} = \quad \Delta^6 =$

```
q^6 + q^7 + 2*q^8 + 3*q^9 + 3*q^11 + 2*q^12 + 2*q^13
+ q^16 + 4*q^17 + q^18 + 4*q^19 + 2*q^22 + 4*q^24 +
3*q^26 + 3*q^27 + 3*q^28 + 3*q^29 + 4*q^31 + 4*q^32
+ 4*q^33 + 4*q^34 + q^36 + q^37 + 4*q^38 + 3*q^39 +
4*q^41 + q^42 + 4*q^44 + 4*q^46 + 4*q^48 + 4*q^49 +
q^51 + 2*q^53 + 4*q^54 + 3*q^56 + 4*q^58 + q^62 +
4*q^63 + 3*q^64 + 3*q^66 + 4*q^67 + 3*q^68 + q^69 +
2*q^72 + 4*q^73 + q^74 + q^76 + 4*q^77 + 3*q^78 +
4*q^79 + q^82 + 3*q^84 + 2*q^86 + q^87 + 4*q^88 +
4*q^89 + 3*q^91 + q^92 + 2*q^93 + 4*q^94 + 3*q^96 +
3*q^97 + q^98 + 2*q^99 + O(q^100) mod 5
```

$f_{4/1,1} = \quad \Delta^8 =$

```
q^8 + 3*q^9 + 4*q^10 + 2*q^11 + q^12 + 4*q^13 + 4*q^14
+ 3*q^15 + 2*q^16 + q^19 + 3*q^21 + 4*q^22 + 2*q^24
+ 4*q^26 + 4*q^27 + 4*q^28 + 4*q^29 + 3*q^31 +
4*q^33 + q^34 + 4*q^35 + 3*q^37 + q^38 + 2*q^39 +
q^43 + 3*q^44 + 2*q^47 + 4*q^51 + 2*q^52 + q^53 +
3*q^54 + q^56 + q^57 + 3*q^58 + 2*q^59 + 4*q^60 +
4*q^61 + 2*q^63 + 3*q^65 + 2*q^66 + q^67 + 4*q^68 +
2*q^69 + 2*q^71 + q^73 + q^74 + 2*q^76 + 2*q^78 +
3*q^79 + 2*q^81 + 3*q^82 + 4*q^85 + 4*q^86 + q^87 +
q^89 + 3*q^90 + q^91 + 3*q^92 + 3*q^93 + 3*q^94 +
4*q^97 + 3*q^98 + 4*q^99 + O(q^100) mod 5
```

$$f_{5/5,1} = \Delta^{10} =$$

```
q^10 + 2*q^15 + q^35 + q^60 + 2*q^65 + q^85 + 2*q^90
+ O(q^100) mod 5
```

$$f_{25/29,1} = \Delta^{50} + 4\Delta^{42}E_4^{24} + 3\Delta^{41}E_4^{27} =$$

```
3*q^41 + 2*q^42 + 4*q^43 + 4*q^44 + 3*q^47 + 2*q^48 +
3*q^49 + q^50 + q^51 + q^52 + 2*q^54 + q^56 + 4*q^58
+ q^59 + 4*q^61 + 4*q^62 + q^63 + 3*q^64 + q^66 +
4*q^67 + 3*q^68 + 3*q^69 + q^71 + q^74 + 2*q^75 +
2*q^76 + 3*q^78 + 4*q^79 + 2*q^81 + 3*q^82 + 2*q^83
+ 4*q^84 + 2*q^88 + 3*q^89 + 4*q^91 + q^92 + 2*q^94
+ 2*q^96 + q^98 + q^102 + q^104 + 4*q^106 + 3*q^107
+ 3*q^108 + 2*q^109 + 4*q^111 + 4*q^112 + 4*q^114 +
3*q^116 + 2*q^118 + 2*q^119 + q^121 + 4*q^122 +
3*q^123 + q^124 + q^126 + 2*q^127 + q^129 + 4*q^132
+ q^134 + 4*q^136 + 4*q^138 + q^139 + q^141 +
3*q^143 + q^144 + q^147 + 3*q^149 + O(q^150) mod 5
```

$$f_{25/5,2} = \Delta^{50} =$$

```
q^50 + 10*q^55 + 15*q^60 + 5*q^65 + 5*q^70 + 12*q^75 +
15*q^80 + 20*q^85 + 10*q^90 + 5*q^95 + 15*q^100 +
10*q^105 + 20*q^110 + 5*q^115 + 20*q^125 + 20*q^135
+ 15*q^140 + 20*q^145 + 10*q^150 + O(q^151) mod 25
```

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