Meta Distributions—Part 1: Definition and Examples

Martin Haenggi, *Fellow, IEEE*

(Invited Paper)

***Abstract*—Meta distributions (MDs) have emerged as a powerful tool in the analysis of wireless networks. Compared to standard distributions, they enable a clean separation of the different sources of randomness, resulting in sharper, more refined results. In particular, they capture the disparity of the performances of individual links or users.**

In this first part of a two-letter series, we start from first principles and give the formal definition of MDs and present several simple yet illustrative examples. Part 2 [1] explores the properties of the MD in more depth and offers multiple interpretations and applications.

***Index Terms*—Meta distributions, wireless networks, stochastic geometry, point processes, signal fraction, interference.**

I. Introduction

When making the distinction between the average (or mean) of a random variable Z and its distribution, it is important to note that the distribution is, in fact, also an average, namely that of the indicator $\mathbf{1}(Z > z)$—assuming we focus on the complementary cumulative distribution function (ccdf). So the average is a mean with 0 parameters (a scalar function of only the distribution), while the distribution is a mean with one parameter. Now, if Z is an “atomic” random variable in the sense that it does not depend on any other source of randomness, then $\mathbb{E}\mathbf{1}(Z > z)$ gives the complete information about all statistics of Z, *i.e.*, the probability of any event can be expressed by adding or subtracting such elementary probabilities.

However, if Z is a function of other sources of randomness, then $\mathbb{E}\mathbf{1}(Z > z)$ alone does not reveal how the statistics of Z depend on the individual random elements. In general Z may depend on many, possibly infinitely many, random variables and random elements (*e.g.*, point processes), such as the signal-to-interference ratio (SIR) in a wireless network. To obtain fine-grained statistical information on how the fading or the point process affects the SIR, we cannot lump all randomness together and consider just the SIR distribution. Instead, we need to dissect the different sources of randomness and analyze their effect on the SIR individually.

To show how this can be achieved, we first focus on the case where $Z = f(X, Y)$ for two random variables X and Y. Throughout the document, we use F for cdfs and $\bar{F}$ for ccdfs.

The two-parameter expectations

$$\begin{aligned}\bar{F}_{[\![Z|Y]\!]}(z, x) &\triangleq \mathbb{E}\mathbf{1}\big(\mathbb{E}[\mathbf{1}(Z > z) \mid Y] > x\big)\\ &= \mathbb{E}\mathbf{1}\big(\mathbb{E}_X\mathbf{1}(Z > z) > x\big)\end{aligned}$$

Martin Haenggi is with the Dept. of Electrical Engineering, University of Notre Dame, Indiana, USA. E-mail: mhaenggi@nd.edu. This work was supported in part by the US National Science Foundation through Grant 2007498.

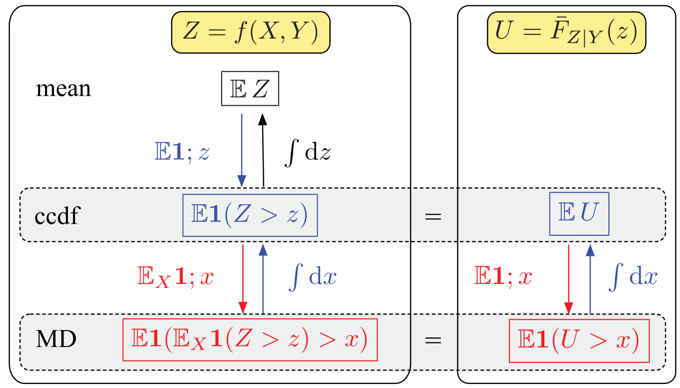

Fig. 1. The left box shows the relationships between the mean, distribution (ccdf), and meta distribution $\bar{F}_{[\![Z|Y]\!]}$ of a random variable $Z = f(X, Y)$. The right box shows that if U is the conditional ccdf given Y, then the ccdf of Z is the mean of U, while the MD of Z is the ccdf of U.

$$\begin{aligned}\bar{F}_{[\![Z|X]\!]}(z, x) &\triangleq \mathbb{E}\mathbf{1}\big(\mathbb{E}[\mathbf{1}(Z > z) \mid X] > x\big)\\ &= \mathbb{E}\mathbf{1}(\mathbb{E}_Y\mathbf{1}(Z > z) > x),\end{aligned}$$

where $\mathbb{E}_X$ denotes the expectation with respect to X, are natural extensions of the standard ccdfs. They are distributions of conditional distributions, *i.e.*, *meta distributions*. The double brackets in the subscript $[\![Z \mid Y]\!]$ indicate that the function is a meta distribution (MD), and $Z \mid Y$ indicates that it is the MD of Z where the conditional distribution is given Y.

So, going from left to right in the chain

$$\mathbb{E}Z \;\to\; \mathbb{E}\mathbf{1}(Z > z) \;\to\; \begin{cases}\mathbb{E}\mathbf{1}(\mathbb{E}_X\mathbf{1}(Z > z) > x)\\ \mathbb{E}\mathbf{1}(\mathbb{E}_Y\mathbf{1}(Z > z) > x),\end{cases}$$

we have 0, 1, and 2 parameters. Parameters can be re-eliminated by integration—by integrating the MD over x we obtain the ccdf, and by further integrating over z, we obtain the mean. These relationships are summarized in Fig. 1. It is apparent that the step from the ccdf to the MD is analogous to that from the mean to the ccdf, *i.e.*, the MD provides refined information compared to the ccdf in the same way the ccdf provides refined information compared to the mean. Indeed, if we focus on the random variables (conditional probabilities)

$$U = U(z, Y) = \mathbb{E}\mathbf{1}(Z > z \mid Y) = \bar{F}_{Z|Y}(z), \tag{1}$$

we recognize that the ccdf of Z is the mean of U while the MD is the ccdf of U, shown on the right in Fig. 1.

Accordingly, defining the moments of U by

$$M_b(z) \triangleq \mathbb{E}(U^b) = b\int_0^1 x^{b-1}\bar{F}_{[\![Z|Y]\!]}(z, x)\mathrm{d}x, \quad b \in \mathbb{C},$$

we have $M_1(z) = \bar{F}_Z(z)$; the higher moments of U are revealed only in the MD.

What if we integrate the MD over z first? Does the resulting function of x have any significance? This question is addressed in the companion paper [1].

The case $Z = f(X,Y)$ is easily extended to $Z = f(\mathcal{X},\mathcal{Y})$, where $\mathcal{X}$, $\mathcal{Y}$ form a partition[1] of a random vector $(X_1,\ldots,X_m)$, $m \in \mathbb{N}\cup\{\infty\}$. We formally define the MD for this general case.

Definition 1 (Meta distribution) *Let $Z = f(\mathcal{X},\mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ form a partition of all random elements that Z depends on. The meta distribution of Z given $\mathcal{Y}$ is*

$$\begin{aligned}\bar{F}_{[\![Z|\mathcal{Y}]\!]}(z,x) &\triangleq \mathbb{E}\mathbf{1}(\mathbb{E}[\mathbf{1}(Z>z)\mid\mathcal{Y}]>x)\\ &= \mathbb{E}\mathbf{1}(\mathbb{E}_{\mathcal{X}}\mathbf{1}(Z>z)>x),\end{aligned} \tag{2}$$

and the meta distribution of Z given $\mathcal{X}$ is

$$\begin{aligned}\bar{F}_{[\![Z|\mathcal{X}]\!]}(z,x) &\triangleq \mathbb{E}\mathbf{1}(\mathbb{E}[\mathbf{1}(Z>z)\mid\mathcal{X}]>x)\\ &= \mathbb{E}\mathbf{1}(\mathbb{E}_{\mathcal{Y}}\mathbf{1}(Z>z)>x).\end{aligned} \tag{3}$$

MDs can equivalently be expressed as

$$\begin{aligned}\bar{F}_{[\![Z|\mathcal{Y}]\!]}(z,x) &= \mathbb{P}(\mathbb{P}(Z>z\mid\mathcal{Y})>x)\\ &= \mathbb{P}(\mathbb{P}_{\mathcal{X}}(Z>z)>x).\end{aligned} \tag{4}$$

In stochastic geometry applications of the MD, we often set $\mathcal{Y} = \Phi$ for a point process Φ, and $\mathcal{X} = (h_1,h_2,\ldots)$ is the vector of fading random variables. This way, the MD achieves a *time scale decomposition* by separating the small-scale randomness and the spatial randomness. Time averages (over small-scale fading) and spatial averages (averaging over the point process) are taken in two steps. In contrast, in standard distributions such as the SIR ccdf $\mathbb{P}(\mathsf{SIR}>z)$ all randomness is eliminated in one step, irrespective of its source and nature, which masks any insight into the individual effects of spatial and temporal randomness.

In the next section, we present several examples of meta distributions. For simplicity and tractability, we focus on the case $Z = f(X,Y)$, where $f \geq 0$ and strictly monotone in both arguments and X and Y are independent and non-negative.

II. Examples

In this section, we use the shortcuts $U \triangleq \bar{F}_{Z|Y}(z) = \mathbb{E}\mathbf{1}(Z>z\mid Y)$ and $V \triangleq \bar{F}_{Z|X}(z) = \mathbb{E}\mathbf{1}(Z>z\mid X)$. These random variables are the conditional distributions of Z given Y and X, respectively. Their distributions are the MDs $\mathbb{P}(U>x) \equiv \bar{F}_{[\![Z|Y]\!]}(z,x)$ and $\mathbb{P}(V>x) \equiv \bar{F}_{[\![Z|X]\!]}(z,x)$.

A. Ratio of Exponential Random Variables

Let X and Y be independent exponential random variables with means 1 and $1/\mu$, respectively. Define $Z \triangleq X/Y$.

The ccdf of Z is

$$\bar{F}_Z(z) = \frac{\mu}{z+\mu}.$$

In this case, $\mathbb{E}Z$ does not exist. The conditional ccdf given Y is the random variable

$$U \triangleq \bar{F}_{Z|Y}(z) = \mathbb{E}\mathbf{1}(Z>z\mid Y) = e^{-Yz},$$

supported on $[0,1]$.

The distribution of U is the MD $\bar{F}_{[\![Z|Y]\!]}$, obtained as

$$\begin{aligned}\mathbb{P}(U>x) &= \mathbb{P}(e^{-Yz}>x) \qquad (5)\\ &= \mathbb{P}(Y\leq -\log(x)/z)\\ &= 1-x^{\mu/z}. \qquad (6)\end{aligned}$$

The ccdf of Z is

$$\mathbb{E}U = \int_0^1 (1-x^{\mu/z})\mathrm{d}x = \frac{\mu}{\mu+z},$$

and the moments are

$$\mathbb{E}(U^b) = \frac{\mu}{\mu+bz}.$$

The variance follows as

$$\operatorname{var} U = \frac{\mu z^2}{(\mu+2z)(\mu+z)^2}.$$

Interestingly, the variance is maximized when μ and z have the golden ratio $z = (\sqrt{5}+1)\mu/2$ where it has the fixed value 0.09.

For the "reverse" MD $\bar{F}_{[\![Z|X]\!]}$, we have

$$V \triangleq \bar{F}_{Z|X}(z) = 1-e^{-\mu X/z} = \mathbb{E}\mathbf{1}(Z>z\mid X),$$

and the distribution of V is the MD given as

$$\begin{aligned}\mathbb{P}(V>x) &= \mathbb{P}(1-e^{-\mu X/z}>x)\\ &= \mathbb{P}(\mu X/z > -\log(1-x))\\ &= (1-x)^{z/\mu}.\end{aligned} \tag{7}$$

It is apparent that for fixed z, (6) and (7) are inverses. In Part 2 [1, Cor. 2] we will present a sufficient condition for this property to hold.

The MD $\bar{F}_{[\![Z|Y]\!]}$ has relevance in uplink cellular networks, where BSs form a Poisson point process Φ of intensity λ. Users are served by the nearest BS, and the channel is subject to power-law path loss with exponent $\alpha = 2$ and Rayleigh fading. The received signal power from a user at an arbitrary location is $S = h/R^2$, where the distance R is Rayleigh distributed with mean $1/(2\sqrt{\lambda})$ and h is the fading random variable. Since R^2 is exponential with mean $1/(\lambda\pi)$, we obtain from (6)

$$\bar{F}_{[\![S|\Phi]\!]}(z,x) = \bar{F}_{[\![Z|Y]\!]}(z,x) = 1-x^{\lambda\pi/z}. \tag{8}$$

If the users form a stationary and ergodic point process and S_u is the received signal power from user u, $1-x^{\lambda\pi/z}$ is the fraction of users for which $S_u > z$ with probability at least x, for each realization of Φ. Fig. 2 shows a realization of this network for $\lambda = 1$, with the individual probabilities $\mathbb{P}(S_u > 1 \mid \Phi)$ for each user. The conventional signal strength analysis $\mathbb{P}(S>z)$ involves a sweeping average over all randomness and thus only reveals the global average of these reliabilities, which is $\lambda\pi/(\lambda\pi+z) = \pi/(\pi+1)$. In contrast, the MD analysis divulges the entire distribution of

[1]By a (two-element) partition of a vector in $\mathbb{R}^m$, we mean that $\mathcal{X} = (X_i)_{i\in\mathcal{P}_1}$ and $\mathcal{Y} = (X_i)_{i\in\mathcal{P}_2}$, where $\{\mathcal{P}_1,\mathcal{P}_2\}$ is a partition of $[m]$, where $[m] = \{1,\ldots,m\}$.

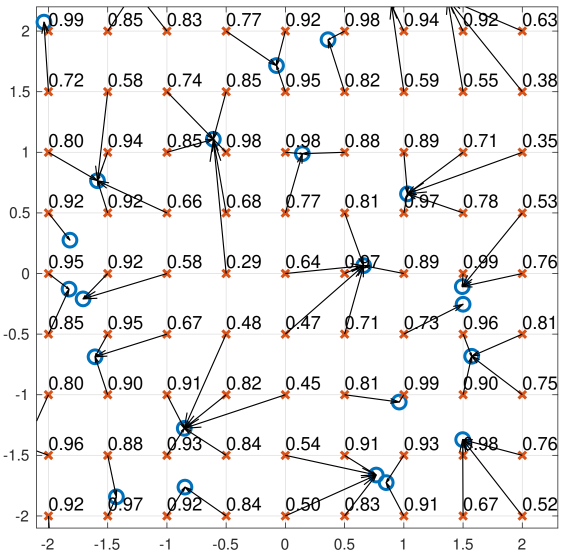

Fig. 2. Realization of cellular uplink network where BSs, marked by blue circles ○, form a PPP of intensity $\lambda = 1$ and users a square lattice of density 4, marked by red crosses ×. Channels are subject to Rayleigh fading and path loss with exponent 2. The arrow indicates the nearest BS to each user u, and the number is the reliability of the up-link $\mathbb{P}(S_u > 1 \mid \Phi)$.

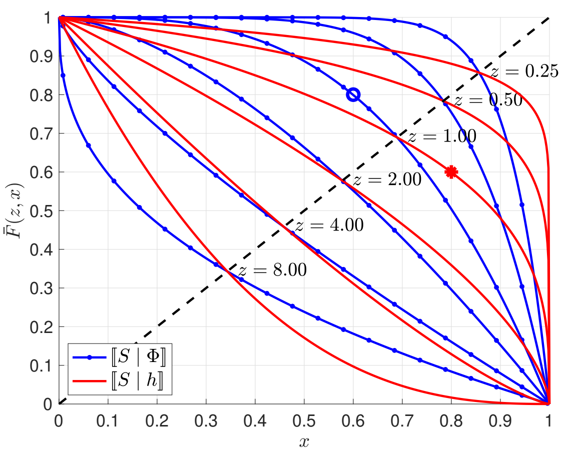

Fig. 3. Meta distributions $\bar{F}_{[\![S|\Phi]\!]}(z, x)$ (blue) and $\bar{F}_{[\![S|h]\!]}(z, x)$ (red) for $S = h/Y$ where h is exponential with mean 1 and Y is exponential with mean $1/\pi$. For instance, for $z = 1$ and fixed positions and fading, the blue circle ○ shows that 80% of the users achieve $S = 1$ with probability at least 0.6. The corresponding red star $*$ shows that with high mobility and fixed random transmit power (but no fading), 60% of the users achieve $S = 1$ with probability at least 0.8.

the probabilities given in (8). A histogram of the numbers in the figure approaches the probability density function (pdf) of U, *i.e.*, the derivative of the MD w.r.t. x, given by

$$f_{[\![Z|Y]\!]}(z, x) = \frac{\lambda\pi}{z} x^{\lambda\pi/z - 1},$$

which we may refer to as the *meta pdf*. This density function shows how concentrated or disparate the conditional probabilities of the users are for a given threshold z.

How about the reverse MD $\bar{F}_{[\![Z|X]\!]} = \bar{F}_{[\![S|h]\!]}$? With some imagination we can think of a scenario where each user selects an exponentially distributed transmit power h with mean 1 that is fixed over time, and users or base stations are highly mobile, such that the link distances from one transmission to the next are iid. In this case, Y is varying over time while X is fixed, hence it is natural to condition on X. V is the time-averaged signal power distribution.

Fig. 3 shows cross-sections of the MDs for $\lambda = 1$ for several values of z. It is apparent that $\bar{F}_{[\![S|\Phi]\!]}(\cdot, z)$ and $\bar{F}_{[\![S|h]\!]}(\cdot, z)$ are inverses of each other.

B. Ratio of Exponential and Weibull Random Variables

To capture path loss exponents $\alpha \neq 2$ in the previous example, we generalize the distribution of Y to the Weibull distribution

$$\bar{F}_Y(x) = e^{-\lambda\pi x^\delta},$$

where $\delta = 2/\alpha$. Its mean is $\Gamma(1 + 1/\delta)/(\lambda\pi)^{1/\delta}$. We obtain

$$\begin{aligned}\bar{F}_{[\![Z|Y]\!]}(z, x) &= \mathbb{P}(U > x) = \mathbb{P}(Y \leq -\log(x)/z) \\ &= 1 - \exp\left(-\lambda\pi\left(\frac{-\log x}{z}\right)^\delta\right)\end{aligned}$$

and

$$\begin{aligned}\bar{F}_{[\![Z|X]\!]}(z, x) &= \mathbb{P}(V > x) = \mathbb{P}(1 - e^{-\lambda\pi(X/z)^\delta} > x) \\ &= \exp\left(-z\left(\frac{-\log(1-x)}{\lambda\pi}\right)^{1/\delta}\right).\end{aligned}$$

Considering a fractional power control scheme that results in a received power $S = hR^{\alpha\varepsilon}R^{-\alpha}$, $\varepsilon \in (0, 1]$, we observe that power control is not always beneficial. It yields an effective path loss exponent $\alpha(1 - \varepsilon)$ but decreasing the path loss (increasing $\delta = 2/(\alpha(1-\varepsilon))$) only improves $\bar{F}_{[\![Z|Y]\!]}(z, x)$ for $x < e^{-z}$. For higher target reliabilities or high thresholds z, $\varepsilon = 0$ (no power control) is best. The reason is that for large z or high reliabilities, the successful links are mostly those with distances smaller than 1. Full path loss inversion, *i.e.*, $\epsilon \to 1$, implies $Y \equiv 1$ and (5) shows that the MD approaches the step function $\mathbb{P}(e^{-z} > x) = 1 - u(x - e^{-z})$, where $u(x) = \mathbf{1}(x \geq 0)$. In contrast, $\varepsilon \downarrow -\infty$ implies infinite path loss, which means S is infinite if $R < 1$ and 0 if $R > 1$. The resulting MD is, for all x and z, the probability that there is a point within distance 1 in the PPP, *i.e.*, $1 - e^{-\lambda\pi}$.

C. Sum of Exponential Random Variables

As in the first example, we let X and Y be independent exponential random variables with means 1 and $1/\mu$, respectively. Here we consider the sum $Z \triangleq X + Y$. We have

$$\bar{F}_Z(z) = \frac{\mu e^{-z} - e^{-\mu z}}{\mu - 1}, \quad \mu \neq 1,$$

and

$$\bar{F}_Z(z) = e^{-z}(z + 1), \quad \mu = 1.$$

The conditional ccdf is

$$U \triangleq \bar{F}_{Z|Y}(z) = \begin{cases} e^{-(z-Y)} & Y < z \\ 1 & Y \geq z. \end{cases}$$

If $Y > z$, which happens with probability $e^{-\mu z}$, $\mathbb{P}(U > x) = \mathbf{1}(x < 1)$. If $Y < z$ and $x < 1$,

$$\mathbb{P}(U > x) = \mathbb{P}(e^{-(z-Y)} > x) = \begin{cases} e^{-\mu(\log x + z)} = x^{-\mu} e^{-\mu z}, & x > e^{-z} \\ 1, & x \le e^{-z}. \end{cases}$$

Integration over x yields

$$\mathbb{E}(U) = e^{-z} + e^{-\mu z} \int_{e^{-z}}^{1} x^{-\mu} \mathrm{d}x = \bar{F}_Z(z).$$

For the "reverse" MD we consider the conditional ccdf

$$V \triangleq F_{Z|X}(z) = \begin{cases} e^{-\mu(z-X)} & X < z \\ 1 & X \ge z. \end{cases}$$

If $X > z$, which happens with probability e^{-z}, $\mathbb{P}(V > x) = \mathbf{1}(x < 1)$. If $X < z$ and $x < 1$,

$$\begin{aligned} \mathbb{P}(V > x) &= \mathbb{P}(e^{-\mu(z-X)} > x) \\ &= \mathbb{P}\left(X > \frac{\log x}{\mu} + z\right) \\ &= \begin{cases} e^{-(\log(x)/\mu + z)} = x^{-1/\mu} e^{-z}, & x > e^{-z\mu} \\ 1, & x \le e^{-z\mu}. \end{cases} \end{aligned}$$

The distribution of Z is recovered from

$$\mathbb{E}(V) = e^{-\mu z} + e^{-z} \int_{e^{-\mu z}}^{1} x^{-1/\mu} \mathrm{d}x = \bar{F}_Z(z).$$

D. Ratio of Exponential and Exponential+Gamma Random Variables

Let $Z \triangleq X/(X+Y)$, where X is exponential with mean 1 and Y is gamma distributed as

$$f_Y(y) = \frac{1}{\Gamma(a)} y^{a-1} e^{-y},$$

and independent of X. The cdf is $F_Y(y) = \bar{\gamma}(a, y)$, where $\bar{\gamma}$ is the normalized lower gamma function[2], and $\mathbb{E}(Y) = a$. Z is known to be beta distributed as

$$f_Z(z) = a(1-z)^{a-1}; \qquad F_Z(z) = (1-z)^a$$

with mean $1/(1+a)$. The conditional ccdf is

$$U \triangleq \bar{F}_{Z|Y}(z) = e^{-zY/(1-z)}.$$

Letting $\zeta = z/(1-z)$,

$$\begin{aligned} \mathbb{P}(U > x) &= \mathbb{P}(e^{-Y\zeta} > x) \\ &= \mathbb{P}(Y \le -\log(x)/\zeta) \\ &= \bar{\gamma}(a, -\log(x)/\zeta). \end{aligned} \tag{9}$$

The ccdf of Z is $\mathbb{E}(U)$, retrieved by integration over x:

$$\int_0^1 \bar{\gamma}(a, -\log(x)/\zeta) \mathrm{d}x = \frac{1}{(1+\zeta)^a} = (1-z)^a$$

[2]In Matlab, $\bar{\gamma}(a, z)$ is calculated using `gammainc(z,a)`. In Maple, it is `1-GAMMA(a,z)/GAMMA(a)`, and in Mathematica, it is `1-Gamma[a,z]/Gamma[a]`.

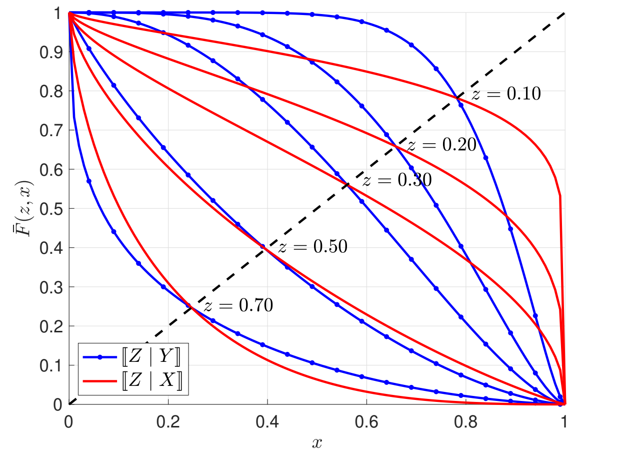

Fig. 4. Meta distributions $\bar{F}_{[\![Z|Y]\!]}(z, x)$ (blue) given in (9) and $\bar{F}_{[\![Z|X]\!]}(z, x)$ (red) given in (10) for $a = 3/2$ and various values of z.

For the "reverse" MD $\bar{F}_{[\![Z|X]\!]}$,

$$V = \mathbb{E}\mathbf{1}(Z > z \mid X) = \bar{\gamma}(a, X(1/z - 1)) = \bar{\gamma}(a, X/\zeta),$$

whose distribution of V is the MD

$$\begin{aligned} \mathbb{P}(V > x) &= \mathbb{P}(\bar{\gamma}(a, X/\zeta) > x) \\ &= \mathbb{P}(X > \zeta \bar{\gamma}_a^{-1}(x)) \\ &= e^{-\zeta \bar{\gamma}_a^{-1}(x)}, \end{aligned} \tag{10}$$

where $\bar{\gamma}_a^{-1}$ is the inverse of the lower incomplete gamma function[3] for given a, *i.e.*, $\bar{\gamma}(a, y) = x \Leftrightarrow y = \bar{\gamma}_a^{-1}(x)$. Hence we have proven that

$$\int_0^1 e^{-\zeta \bar{\gamma}_a^{-1}(x)} \mathrm{d}x = (1+\zeta)^{-a},$$

which is an integral that standard mathematical software cannot solve.

Fig. 4 shows cross-sections of the two MDs $\bar{F}_{[\![Z|Y]\!]}$ and $\bar{F}_{[\![Z|X]\!]}$ for $a = 3/2$. $\bar{F}_{[\![Z|Y]\!]}$ given in (9) is the MD of the signal fraction (SF), defined as $\mathsf{SF} = S/(S+I) = \mathsf{SIR}/(\mathsf{SIR}+1)$ [2], if the signal power is exponential with mean 1 and the interference power is gamma distributed (and not subject to small-scale fading).

Comparing this example with the first two, we conclude that the distributions of the forms $Z = X/Y$ and $Z' = X/(X+Y)$ are related by replacing the parameter z by $z/(1-z)$, *i.e.*, $F_Z(z) = F_{Z'}(z/(z+1))$ and $F_{Z'}(z) = F_Z(z/(1-z))$. This holds more generally whenever Z' can be expressed by an invertible function of Z, and also when X and Y are dependent.

Accordingly, by replacing ζ by z in (9), we obtain the SIR MD for the same scenario (Rayleigh fading in the desired link and gamma distributed interference).

[3]In Matlab, this function is implemented as `gammaincinv(x,a)`. In Mathematica, it is `InverseGammaRegularized[a,1-x]`.

E. Ratio of Exponential and Exponential+Inverse Gamma Random Variables

Let $Z \triangleq X/(X+Y)$, where X is exponential with mean 1 and Y is inverse gamma distributed as

$$f_Y(y) = b^2 y^{-3} e^{-b/y},$$

independent of X. The cdf is $F_Y(y) = 1 - \bar{\gamma}(2, b/y) = (y+b)e^{-b/y}/y$, and $\mathbb{E}(Y) = b$.

As before, the conditional ccdf is $U = e^{-zY/(1-z)}$, and, letting $\zeta = z/(1-z)$,

$$\begin{aligned}\mathbb{P}(U > x) &= \mathbb{P}(Y \leq -\log(x)/\zeta) \\ &= \left(1 - \frac{b\zeta}{\log x}\right) e^{b\zeta/\log x}. \qquad (11)\end{aligned}$$

The ccdf of Z is $\mathbb{E}(U)$, obtained by integration over x, is

$$F_Z(z) = 2c\,\mathrm{K}(1, 2c) + 2c^2\,\mathrm{K}(0, 2c),$$

where K is the modified Bessel function of the second kind[4] and $c = \sqrt{b\zeta} = \sqrt{bz/(1-z)}$. This is an instance where the MD is in closed-form, while the ccdf requires transcendental functions. As before, $\bar{F}_{[\![Z|Y]\!]}$ can be interpreted as an SF MD, in this case for inverse gamma distributed interference.

For $\bar{F}_{[\![Z|X]\!]}$, we let

$$V = \mathbb{E}\mathbf{1}(Z > z \mid X) = F_Y(X/\zeta) = (1 + b\zeta/X)e^{-b\zeta/X}.$$

The distribution of V is the MD given as

$$\begin{aligned}\mathbb{P}(V > x) &= \mathbb{P}\left(X > \frac{-b\zeta}{\mathrm{W}_{-1}(-x/e)+1}\right) \\ &= \exp\left(\frac{b\zeta}{\mathrm{W}_{-1}(-x/e)+1}\right), \quad x < 1, \qquad (12)\end{aligned}$$

where W_{-1} is the -1-st branch of the Lambert W function[5].

F. Ratio of Gamma Random Variables

Lastly, we generalize the result from Subs. II-A to the case where X is gamma distributed with mean 1 and Y is independent and gamma distributed. The cdfs are

$$\begin{aligned}F_X(x) &= \bar{\gamma}(a, ax), \quad a > 0, \\ F_Y(y) &= \bar{\gamma}(c, by), \quad b, c > 0.\end{aligned}$$

By independence, $\mathbb{E}Z = b/(c-1)$ if $c > 1$.

The ccdf of $U = \bar{F}_{Z|Y}(z) = 1 - \bar{\gamma}(a, aYz)$ is

$$\begin{aligned}\mathbb{P}(U > x) &= \mathbb{P}(1 - \bar{\gamma}(a, aYz) > x) \\ &= \mathbb{P}(Y \leq \bar{\gamma}_a^{-1}(1-x)/(az)) \\ &= \bar{\gamma}(c, b\bar{\gamma}_a^{-1}(1-x)/(az)). \qquad (13)\end{aligned}$$

In the reverse MD $\bar{F}_{[\![Z|X]\!]}$, we have $V = \bar{\gamma}(c, bX/z)$ whose ccdf is

$$\begin{aligned}\mathbb{P}(V > x) &= \mathbb{P}(\bar{\gamma}(c, bX/z) > x) \\ &= \mathbb{P}(X > z\bar{\gamma}_c^{-1}(x)/b) \\ &= 1 - \bar{\gamma}(a, az\bar{\gamma}_c^{-1}(x)/b). \qquad (14)\end{aligned}$$

[4]Implemented in Maple and Mathematica as `BesselK(v,x)` and in Matlab as `besselk(v,x)`.

[5]Implemented in Maple as `LambertW(-1,x)`, in Mathematica as `ProductLog[-1,x]`, and in Matlab as `lambertw(-1,x)`. For $x \in [-1/e, 0)$, it is real-valued, with a range from -1 to $-\infty$.

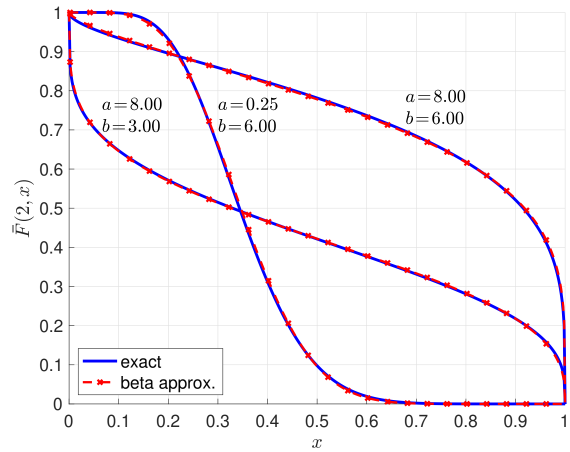

Fig. 5. Meta distributions $\bar{F}_{[\![Z|Y]\!]}(2, x)$ as given in (13) (blue) and beta approximations (red) for $c = 2$ and three pairs of values for a, b.

As in the previous example, it is apparent that (13) and (14) are mutual inverses.

If z in the MDs is replaced by $z/(1-z)$, the MDs of $X/(X+Y)$ are obtained. Hence this example also generalizes the one in Subs. II-D. By setting $a = b = 1$ and $c = a$, we have

$$\mathbb{P}(U > x) = \bar{\gamma}\big(a, \gamma_1^{-1}(1-x)/z\big), \qquad (15)$$

which corresponds to (9) with z replaced by $\zeta = z/(1-z)$ and noting that $\bar{\gamma}_1^{-1}(1-x) = -\log x$.

The approximation of MDs using beta ccdfs is natural and has been frequently used since it was originally proposed in [3]. It is obtained by matching the first and second moments of U to those of a beta distribution. Fig. 5 shows that an excellent approximation is obtained for several qualitatively different MDs (13). These cases where the MD has a closed-form expression are ideal to assess the accuracy of the beta approximation.

III. Concluding Remarks

In this first part of a two-letter series, the concept of meta distributions is introduced and motivated as a natural extension of distributions (ccdfs). We have calculated MDs for several simple examples and shown that ccdfs are a special case of MDs, obtained by integration, in exactly the same way means are obtained by integrating ccdfs. In Part 2 [1], we will explore the properties of the MD in more depth, formally state a sufficient condition for the property that "forward" and "reverse" MDs are inverses of each other, present two applications to Poisson networks, and discuss different interpretations.

References

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