1

On the SIR Meta Distribution for Poisson Networks with Interference Cancellation

Yuanjie Wang, Qimei Cui, Senior Member, IEEE, Martin Haenggi, Fellow, IEEE, and Zhenhui Tan, Member, IEEE

Abstract—This letter presents a theoretical framework for the analysis of the meta distribution of the SIR for Poisson networks with interference cancellation (IC) enabled at the receivers, which gives deep insight into the network performance on a link-wise basis. A simple but insightful IC model named C-IC is studied for which the exact b-th moment of the meta distribution and its beta distribution approximation and some useful bounds are validated. The conditions for the mean local delay to be finite are also stated. The results show that IC improves the performance not only in terms of the mean but also in terms of the variance of the meta distribution.

Index Terms—Stochastic geometry, Poisson networks, meta distribution, interference cancellation, mean local delay.

I. INTRODUCTION

In wireless networks, especially in densely deployed scenarios, one of the key problems is the strong interference experienced at the receivers due to their proximity to the interfering transmitters. Although the network densification greatly enhances the received signal power, the almost equal increase of the interference power limits the improvement of the spectral efficiency. Many works have made tremendous efforts and significant progress in tackling this problem. Roughly speaking, these works can be classified into two categories: one is interference avoidance/coordination, which is interference management through sophisticated spectrum allocation or smart user scheduling algorithms; the other is interference cancellation (IC), which turns to advanced signal processing techniques to cancel some dominant interference signals [1]. From the perspective of spectrum utilization, the former may not be as competitive as the latter, which allows aggressive reuse of the spectrum.

The irregularity of the wireless network topology motivates the application of stochastic geometry as a powerful mathematical tool for network analysis. Several works have already utilized stochastic geometry to study the effect of interference cancellation in wireless networks [2]–[4]. However, these works mainly focus on analyzing performance metrics that are based on the standard success probability, which is obtained

Yuanjie Wang and Zhenhui Tan are with the State Key Laboratory of Railway Traffic Control and Safety, Beijing Jiaotong University, Beijing, 100044, China (e-mail: wang.yuanjie@outlook.com). Qimei Cui is with the National Engineering Laboratory for Mobile Network Technologies, Beijing University of Posts and Telecommunications, Beijing, 100876, China (e-mail: cuiqimei@bupt.edu.cn). Martin Haenggi is with the Dept. of Electrical Engineering, University of Notre Dame, IN, 46556, USA (e-mail: mhaeng-gi@nd.edu). The work was supported by National Natural Science Foundation of China (61471030, 61471058, 61601018), National Science and Technology Major Project of China (2015ZX03001027-003) and the US National Science Foundation (grant CCF 1525904).

by averaging over all the links in a point process. This success probability only quantifies the overall average performance of the entire network, or it can be understood as the performance of the typical receiver, which is an abstracted representative of the receivers in the network obtained by averaging, but it does not offer any information about how the performance of the individual links is distributed. Such fine-grained information is revealed by the meta distribution, which is proposed and defined in [5] as the distribution of the conditional success probability averaged over the fading given the point process. It has been extended to cellular uplink [6], cellular underlaid with D2D [7], and applied to determine the spatial outage capacity [8]. In this letter we combine the meta distribution concept with the IC technique in Poisson networks and present a theoretical framework to allow deep insight into the network performance on a link-wise basis.

II. SYSTEM MODEL

A. Network Model

We consider a Poisson bipolar network in which the transmitters are distributed according to a homogeneous Poisson point process (PPP) $\Phi_{\rm b}$ of intensity $\lambda_{\rm b}$. Each transmitter has a dedicated receiver at distance R_0 in a random orientation. The signals transmitted from the transmitters to the receivers are subject to the standard power law path loss with exponent $\alpha > 2$ and Rayleigh fading with unit mean in the power domain. The fading coefficients are independent across different links. Each transmitter transmits with the same power and has an infinitely backlogged queue such that all transmitters are always in the "on" state. $\lambda_{\rm b}$ is assumed to be high enough so that the network is interference-limited.

B. Interference Cancellation

Close Interference Cancellation (C-IC): This is a commonly used model in the existing literature, see, e.g., [2], [3]. A receiver has a *cancellation region* around itself, in which it is capable of reducing the interference power by $1 - \epsilon$ $(0 \le \epsilon \le 1)$. The cancellation region is a disk with radius $D_c = \sqrt{\frac{n}{\pi \lambda_b}}$ so that on average *n* interfering transmitters lie in the cancellation region. For a receiver *y* with its transmitter *x*, the set of (partially) cancelled interfering transmitters is $C = \Phi_I \bigcap \mathbf{B}(y, D_c)$, where Φ_I is the set of all the potential interfering transmitters to the receiver and $\mathbf{B}(y, D_c)$ is the disk centered at the target receiver with radius D_c .

Such a simple but insightful IC model is based on the following reason:

- Most outages are caused by only a few—often just one strong interferers, which are usually close to the target receivers.
- In practice the number of cancelled interferers of a receiver can be time-varying due to the changes in the propagation environment, hence it makes sense to consider the averaged IC capability.

The residual interference power coefficient ϵ accounts for the imperfect cancellation capability in reality. For instance, the model in consideration can be realized in the form of successive IC (SIC) with only the local CSI of some nearest transmitters at the receiver, then ϵ is understood as a quantity to measure the accuracy of the CSI.

III. THE META DISTRIBUTION WITH INTERFERENCE CANCELLATION

Instead of the standard (mean) success probability $p_{\rm s}(\theta) \triangleq \mathbb{P}^{!x}(\mathsf{SIR}_y > \theta)$ at the receiver y (with its transmitter $x \in \Phi_{\rm b}$), where $\mathbb{P}^{!x}$ denotes the reduced Palm measure [9, Chap. 8] of $\Phi_{\rm b}$ given x, we focus on the conditional success probability $P_{\rm s}(\theta) \triangleq \mathbb{P}(\mathsf{SIR}_y > \theta \mid \Phi_{\rm b})$ given the transmitter process and averaged over the fading. $P_{\rm s}$ is a random variable within [0, 1] with the ccdf (defined as the meta distribution [5]) given by

$$\bar{F}(\theta, t) \triangleq \bar{F}_{P_{s}}(t) = \mathbb{P}^{!x}(P_{s}(\theta) > t), \quad \theta \in \mathbb{R}^{+}, \ t \in [0, 1],$$
(1)

The ergodicity of the PPP gives an alternative expression of $\bar{F}(\theta,t)$, which is

$$\bar{F}(\theta, t) = \lim_{r \to \infty} \frac{1}{\lambda_{\rm b} \pi r^2} \sum_{\substack{\tilde{y} \in \Phi_{\rm b} \\ \|\tilde{y}\| \le r}} \mathbf{1}(\mathbb{P}(\mathsf{SIR}_y > \theta \mid \Phi_{\rm b}) > t),$$

where \tilde{y} is the transmitter of y and $\mathbf{1}(\cdot)$ is the indicator function. This explicitly shows that $\bar{F}(\theta, t)$ is the fraction of receivers that experience an SIR above θ with reliability at least t.

Also, due to the stationarity, the (Palm) distribution [9, Chap. 8] of $P_s(\theta)$ at any receiver is identical. Henceforth, we focus on the receiver at the origin with its transmitter at x, where $||x|| = R_0$.

Then the set of (partially) cancelled interfering transmitters is $C = \Phi_{I} \bigcap \mathbf{B}(o, D_{c})$, where $\Phi_{I} = \Phi_{b} \setminus \{x\}$. For the C-IC model, the SIR at the origin is given by

$$\mathsf{SIR}_{o} = \frac{h_{0}R_{0}^{-\alpha}}{\sum\limits_{x'\in\Phi_{\mathrm{I}}} h_{x'}R_{x'}^{-\alpha}(1-(1-\epsilon)\mathbf{1}(x'\in\mathcal{C}))}, \qquad (2)$$

where h_0 is the Rayleigh fading experienced by the intended signal, $h_{x'}$ and $R_{x'} = ||x'||$ are the fading and distance of the link from $x' \in \Phi_I$ to the origin, respectively.

Theorem 1 (Moments of $P_s(\theta)$ of C-IC) The b-th moment $M_b, b \in \mathbb{C}$, of the conditional success probability $P_s(\theta)$ for Poisson networks with C-IC is given by

$$M_{b} = \exp(-C\epsilon^{\delta}\theta^{\delta}\Gamma(b+\delta)/\Gamma(b))$$

$$\cdot \exp(c_{2}F_{1}(b,-\delta;1-\delta;-\theta\epsilon(R_{0}/D_{c})^{\alpha}))$$

$$\cdot \exp(-c_{2}F_{1}(b,-\delta;1-\delta;-\theta(R_{0}/D_{c})^{\alpha})), \quad (3)$$

where $\delta = 2/\alpha$, $C = \lambda_{\rm b} \pi R_0^2 \Gamma(1-\delta)$, $c = \lambda_{\rm b} \pi D_{\rm c}^2$, and $_2F_1$ is the Gaussian hypergeometric function.

Proof: For the C-IC model, the conditional success probability is

$$P_{s}(\theta) = \mathbb{P}\left(\mathsf{SIR}_{o} > \theta \mid \Phi_{b}\right)$$

$$= \mathbb{P}\left(h_{0} > \theta R_{0}^{\alpha} \sum_{x \in \Phi_{I}} h_{x} R_{x}^{-\alpha} g(x) \mid \Phi_{b}\right)$$

$$= \mathbb{P}\left(h_{0} > \theta R_{0}^{\alpha} \left(\sum_{x \in \mathcal{C}} \epsilon h_{x} R_{x}^{-\alpha} + \sum_{x \in \Phi_{I} \setminus \mathcal{C}} h_{x} R_{x}^{-\alpha}\right) \mid \Phi_{b}\right)$$

$$= \prod_{x \in \mathcal{C}} \frac{1}{1 + \epsilon \theta \frac{R_{x}^{-\alpha}}{R_{0}^{-\alpha}}} \cdot \prod_{x \in \Phi_{I} \setminus \mathcal{C}} \frac{1}{1 + \theta \frac{R_{x}^{-\alpha}}{R_{0}^{-\alpha}}}.$$
(4)

 M_b follows as

$$M_{b} = \mathbb{E}\left(\prod_{x \in \mathcal{C}} \frac{1}{\left(1 + \epsilon \theta \frac{R_{x}^{-\alpha}}{R_{0}^{-\alpha}}\right)^{b}} \cdot \prod_{x \in \Phi_{I} \setminus \mathcal{C}} \frac{1}{\left(1 + \theta \frac{R_{x}^{-\alpha}}{R_{0}^{-\alpha}}\right)^{b}}\right)$$
$$= \mathbb{E}\left(\prod_{x \in \mathcal{C}} \frac{1}{\left(1 + \epsilon \theta \frac{R_{x}^{-\alpha}}{R_{0}^{-\alpha}}\right)^{b}}\right) \cdot \mathbb{E}\left(\prod_{x \in \Phi_{I} \setminus \mathcal{C}} \frac{1}{\left(1 + \theta \frac{R_{x}^{-\alpha}}{R_{0}^{-\alpha}}\right)^{b}}\right)$$
$$= \exp\left(-\int_{0}^{D_{c}} 2\pi\lambda_{b}\left(1 - \left(1 + \epsilon \theta R_{0}^{\alpha}r^{-\alpha}\right)^{-b}\right)rdr\right)$$
$$\cdot \exp\left(-\int_{D_{c}}^{\infty} 2\pi\lambda_{b}\left(1 - \left(1 + \theta R_{0}^{\alpha}r^{-\alpha}\right)^{-b}\right)rdr\right)$$
(5)

where the second equality is due to the fact that the two disjoint subsets $\Phi \cap \mathbf{B}(o, D_c)$ and $\Phi \setminus \mathbf{B}(o, D_c)$ form two independent PPPs of the same intensity λ_b . The third equality uses the probability generating functional (PGFL) of the PPP [9, Chap. 4]. The integrals can be simplified to the Gaussian hypergeometric function form in the same way as in [5].

With the *b*-th moment, the meta distribution can be calculated through the Gil-Pelaez theorem [10], which is based on the purely imaginary moments $M_{jt}, t \in \mathbb{R}, j \triangleq \sqrt{-1}$. To avoid the calculation of higher-order imaginary moments, we propose the approximation of the meta distribution by the beta distribution through matching the first and second moments, as suggested in [5].

Corollary 1 (Beta approximation for the meta distribution) *Through matching the first and second moments, the meta distribution of C-IC is approximated by the beta distribution as,*

$$\bar{F}(\theta,t) \approx 1 - I_x \left(\frac{\beta M_1}{1 - M_1}, \beta\right), \ x \in [0,1], \tag{6}$$

where $\beta = \frac{(M_1 - M_2)(1 - M_1)}{M_2 - M_1^2}$, M_1 and M_2 can be calculated from (5), $I_x(u, v) \triangleq \frac{\int_0^x t^{u-1}(1-t)^{b-1} dt}{B(u,v)}$ is the regularized incomplete beta function and B(u, v) is the beta function.

Next we focus on M_{-1} , which is an important metric named *mean local delay* that quantifies the mean number of transmission attempts needed until the first success if the transmitter is allowed to keep transmitting [11]. **Corollary 2** (Mean local delay of C-IC) In Poisson bipolar networks with C-IC, the necessary and sufficient condition for a receiver to have finite mean local delay is $\epsilon = 0$ and $D_c > 0$.

Proof: Given R_0 and θ , setting b = -1 in (3), it is easy to see that the first factor is finite only if $\epsilon = 0$. Then by using $_2F_1(-1,b;c;z) \equiv 1 - \frac{bz}{c}$, we get $M_{-1} = \exp\left(\lambda_b \pi \theta R_0^{\alpha} \frac{\delta}{1-\delta} D_c^{2-\alpha}\right)$. Since $\alpha > 2$, for finite M_{-1} , D_c needs to be positive.

Remark 1: The finite mean local delay of C-IC is due to the fact that perfect C-IC removes the possibility of the potential interferers being arbitrarily close to the target receiver. This indicates that the effect of perfect IC inside a disk is the same as preventing any interferers to be present in the disk.

Next we derive some bounds to simplify the analysis of the meta distribution of C-IC in the practical regime of the SIR.

Lemma 1 Letting $\theta' = \theta(R_0/D_c)^{\alpha}$ and $z = \frac{\theta'}{1+\theta'}$, for $b \in \mathbb{N}$ we have

$$S_b \triangleq {}_2F_1(b, -\delta; 1-\delta; -\theta') = \sum_{m=0}^{\infty} a_m(b) z^m, \qquad (7)$$

with

$$a_m(b) = \begin{cases} \sum_{i=0}^m {\binom{b}{b-i}} \frac{(-1)^i (b)_{m-i}}{(1-\delta)_{m-i}} & m \le b-1, \\ \frac{(m-1)! (\delta)_b}{(b-1)! (1-\delta)_m} & m > b, \end{cases}$$
(8)

where $\binom{m}{k}$ is the binomial coefficient and $(q)_m \equiv \frac{\Gamma(q+m)}{\Gamma(q)}$ is the Pochhammer function (rising factorial).

Proof:

$${}_{2}F_{1}(b, -\delta; 1-\delta; -\theta') \stackrel{(a)}{=} \frac{{}_{2}F_{1}(b, 1; 1-\delta; \theta'/(1+\theta'))}{(1+\theta')^{b}},$$
$$\stackrel{(b)}{=} (1-z)^{b} \sum_{m=0}^{\infty} \frac{(b)_{m}}{(1-\delta)_{m}} z^{m}, \quad (9)$$

where (a) uses Euler's transformation and (b) uses the series form of the Gaussian hypergeometric function $_2F_1$. By further expanding (9) in z for $b \in \mathbb{N}$, we get the coefficient given by (8).

Since 0 < z < 1 and for all m > b, $m \in \mathbb{N}$, $a_m(b) > 0$, $a_{m+1}(b) = a_m(b) \frac{m}{m+(1-\delta)} < a_m(b)$, Lemma 1 offers a convenient way to get useful bounds for M_b by truncating the series of z.

Proposition 1 (Bounds of M_b **for C-IC)** For the C-IC model under perfect IC (i.e., $\epsilon = 0$), upper and lower bounds of M_b are given by

$$M_b < \hat{M}_b^{(m)} = \exp\left(c(1 - \check{S}_b^{(m)})\right),$$
 (10)

$$M_b > \check{M}_b^{(m)} = \exp\left(c(1 - \hat{S}_b^{(m)})\right).$$
 (11)

where the superscript m refers to truncating the series in (7) up to z^m , i.e.,

$$\check{S}_{b}^{(m)} = 1 + a_{1}(b)z + \dots + a_{m}(b)z^{m},$$
(12)

$$\hat{S}_{b}^{(m)} = 1 + a_{1}(b)z + \dots + a_{m}(b)z^{m} + a_{m+1}(b) \cdot \frac{z^{m+1}}{1-z}.$$
 (13)

Proposition 2 (Approximation of the variance for C-IC) For the C-IC model under perfect IC (i.e., $\epsilon = 0$), the variance $V = M_2 - M_1^2$ of the meta distribution can be approximated by

$$\tilde{V} = c \frac{\delta}{2-\delta} z^2 + \frac{2(c\delta - c\delta^2 - 3c^2\delta^2 + c^2\delta^3)}{(1-\delta)(2-\delta)(3-\delta)} z^3, \quad (14)$$

and for $z \to 0$, we have $V \sim \tilde{V}$, where "~" indicates $V(z)/\tilde{V}(z) \to 1$ as $z \to 0$.

Proof: This can be easily obtained through the Maclaurin expansion of $V = M_2 - M_1^2$ in z and omitting the terms of z with order higher than 3.

We can also obtain useful bounds (or approximation) for the variance by directly applying the bounds in Prop. 2 to M_1 and M_2 , as will be shown in Section IV.

Corollary 3 (Asymptotic property of M_b) For the C-IC model, as $\epsilon \to 0$,

$$M_b \sim e^{-cF(\theta')} (1 - C_b \epsilon^{\delta} \theta^{\delta} + (\epsilon c b \delta z) / (1 - \delta)), \qquad (15)$$

where $F(\theta') = {}_2F_1(b, -\delta; 1 - \delta; -\theta') - 1$, $C_b = C\Gamma(b + \delta)/\Gamma(b)$ and $c = \lambda_b \pi D_c^2$.

Proof: When $\epsilon \to 0$, the first exponential term in (3) can be expanded by $e^{-x} = 1 - x + \frac{x^2}{2} + o(x^2)$. The second and third exponentials give $e^{-c({}_2F_1(b, -\delta; 1-\delta; -\theta') - {}_2F_1(b, -\delta; 1-\delta; -\theta'\epsilon)}$, which can be expanded on ϵ and yield $e^{-cF(\theta')}(1 + (\epsilon cb\delta\theta')/(1-\delta) + O(\epsilon^2))$. Taking the product and omitting the higher-order terms yields the final result.

Remark 2: For the above asymptotic expression, if δ is 1/2 or smaller, the term in $\epsilon^{2\delta}$ should be included, which yields

$$M_b \sim e^{-cF(\theta')} \Big(1 - C_b \theta^\delta \epsilon^\delta + \frac{C_b^2}{2} \theta^{2\delta} \epsilon^{2\delta} + \frac{cbz\delta}{1 - \delta} \epsilon \Big).$$
(16)

IV. NUMERICAL RESULTS

This section provides a numerical evaluation of the analytical results for the meta distribution of the C-IC model. For the beta approximation, Monte Carlo simulation results are used for validation. The basic parameters are set as $\lambda_{\rm b} = 1$, $R_0 = 0.5$, $\alpha = 4$ and $\epsilon = 0$ unless otherwise stated. The radius of the guard region is set to $D_{\rm c} = \sqrt{\frac{n}{\pi}}$ with *n* denoting the average number of cancelled interferers.

Fig. 1 shows the mean and variance of the meta distribution, from which we can see that applying interference cancellation has two-fold benefits to the network in the sense that it not only improves the mean success probability of each transmission link but also decreases the performance disparity among different links. As the IC capability enhances, the peak of the variance keeps decreasing and moving towards higher SIR thresholds.

From Fig. 1 we can also infer that the marginal gain gets smaller with the interference cancellation region enlarging. For instance, the gain of M_1 at $\theta = 0$ dB from n = 0 to n = 1 is



Fig. 1. Analytical results for M_1 and the variance $M_2 - M_1^2$ with $\lambda_b \pi D_c^2 = n$.



Fig. 4. Upper and lower bounds of M_2 ((10) and (11)) for C-IC with $\lambda_b \pi D_c^2 = 1$.



Fig. 2. Comparison of the simulated meta distribution and its beta approximation with $\lambda_b \pi D_c^2 = n$.



Fig. 5. Bounds of the variance for C-IC with $\lambda_{\rm b}\pi D_{\rm c}^2=1.$



Fig. 3. Upper and lower bounds of M_1 for C-IC ((10) and (11)) with $\lambda_{\rm b} \pi D_{\rm c}^2 = 1$.



Fig. 6. M_1 and asymptotic bounds for C-IC with $D_c = 0.5$.

over 100% (from 0.29 to 0.59), while the gain drops to 25% from n = 1 to n = 2, and further drops to 11% from n = 2 to n = 3. This confirms the intuition that usually cancelling only a few nearest interferences is enough.

Fig. 2 confirms the accuracy of the beta approximation for the meta distribution of the C-IC model and provide more fine-grained information about how the link-wise success probability is distributed across the network.

Fig. 3 and Fig. 4 show the bounds for M_1 and M_2 of C-IC model through Prop. 1. We can see that for the practical regime of θ , the second-order approximated upper bound is already very accurate, and the lower bounds (m = 3 in (11)) are also very accurate.

Fig. 5 shows the bounds of the variance for C-IC through the application of (10), (11) (with order 3) and (14). For most practical operating regime of θ , the approximations given by the lower bound of M_1 , M_2 and Prop. 2 are very accurate.

Fig. 6 shows the asymptotics for C-IC as $\epsilon \rightarrow 0$. It shows that the accuracy improves with smaller θ .

V. CONCLUSIONS

This letter studied the meta distribution for Poisson bipolar networks with the capability of interference cancellation. The C-IC model was analyzed by deriving the exact *b*-th moment of the meta distribution, and the beta distribution approximation was confirmed to be accurate via simulation. The mean local delay for the C-IC model was also studied and the analysis showed that to keep the mean local delay finite, perfect IC is required ($\epsilon = 0$). Useful bounds for M_b and closed-form approximation for the variance were also provided, which offered a quick way to get insights into the meta distribution for the practical regime of the SIR threshold.

REFERENCES

- G. Boudreau, J. Panicker, N. Guo, R. Chang, N. Wang, and S. Vrzic,, "Interference coordination and cancellation for 4G networks," *IEEE Commun. Mag.*, vol. 47, no. 4, pp. 74–81, Apr. 2009.
- [2] S. P. Weber, J. G. Andrews, X. Yang and G. D. Veciana, "Transmission capacity of wireless ad hoc networks with successive interference cancellation," *IEEE Trans. Inf. Theory*, vol. 53, no. 8, pp. 2799–2814, Aug. 2007.
- [3] J. Lee, J. G. Andrews, and D. Hong, "Spectrum-sharing transmission capacity with interference cancellation," *IEEE Trans. Commun.*, vol. 61, no. 1, pp. 76–86, Oct. 2012.
- [4] X. Zhang and M. Haenggi, "The performance of successive interference cancellation in random wireless networks," *IEEE Trans. Inf. Theory*, vol. 60, pp. 6368–6388, Oct. 2014.
- [5] M. Haenggi, "The meta distribution of the SIR in Poisson bipolar and cellular networks," *IEEE Trans. Wireless Commun.*, vol. 15, no. 44, pp. 2577–2589, Apr. 2016.
- [6] Y. Wang, M. Haenggi and Z. Tan, "The meta distribution of the SIR for cellular networks with power control," *arXiv preprint arXiv:1702.01864*, 2017.
- [7] M. Salehi, A. Mohammadi and M. Haenggi, "Analysis of D2D underlaid cellular networks: SIR meta distribution and mean local delay," *IEEE Trans. Commun.*, vol. 65, no. 7, pp. 2904–2916, Jul. 2017.
- [8] S. S. Kalamkar and M. Haenggi, "Spatial outage capacity of Poisson bipolar networks," in *Proc. IEEE ICC'17*, Paris, France, May 2017.
- [9] M. Haenggi, Stochastic Geometry for Wireless Networks. Cambridge University Press, 2012.
- [10] J. Gil-Pelaez, "Note on the inversion theorem," *Biometrika*, vol. 38, no. 3-4, pp. 481–482, 1951.
- [11] M. Haenggi, "The local delay in Poisson networks," IEEE Trans. Inf. Theory, vol. 59, pp. 1788–1802, Mar. 2013.