The Stable Point Process and its Applications to Wireless Networks

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***Abstract*—What happens if the points of a point process are repeatedly displaced randomly in their Voronoi cells? This paper shows that the resulting point process, called the *stable point process* (SPP), is of repulsive nature and characterized only by its density. In contrast to the existing soft-core models, the SPP is easy to simulate in arbitrary dimensions and thus fills a nagging void in the arsenal of repulsive point processes. Several pertinent applications of the SPP to wireless networks are discussed and demonstrate its relevance and versatility.**

***Index Terms*—Wireless networks, stochastic geometry, point process, soft-core models, interference.**

I. Introduction

Clustered point processes, in particular those of Coxian type, are versatile, easy to define and simulate and fairly tractable. In contrast, the family of repulsive point processes that are tractable and/or easy to simulate is limited. In this paper, to help address this, we introduce and discuss the *stable point process* (SPP). It is obtained by repeatedly perturbing points uniformly at random in their Voronoi cells. If this process is repeated n times, the SPP is obtained in the limit as $n \to \infty$. The term "stable" comes from the fact that further Voronoi perturbations of the SPP no longer change its law.

II. Definitions

Let $\mathcal{U}(B)$ be the uniform distribution on $B \subset \mathbb{R}^d$ and $U(B)$ a point chosen uniformly at random from B, where the volume (Lebesgue measure) $|B|$ is finite, independently of all other randomness.

For a point set $\varphi \subset \mathbb{R}^d$, let $\varphi(B)$ be number of points in B and let V_x, $x \in \varphi$, be the Voronoi cell of x, *i.e.*, $V_x \triangleq \{y \in \mathbb{R}^d \colon \|y - x\| = \|y - \varphi\|\}$. The ball of radius r centered at x is denoted as $b(x, r)$.

Definition 1 (Proper point pattern and point process) *A deterministic set $\varphi \subset \mathbb{R}^d$ is a* proper point pattern *if it is countable and its convex hull is $\mathbb{R}^d$. A point process Φ is called* proper *if the properties hold almost surely (a.s.)*

This definition implies that for a proper point pattern, all Voronoi cells are finite and that its density

$$\lambda \triangleq \lim_{r \to \infty} \frac{\varphi(b(o,r))}{|b(o,r)|}$$

is positive and finite. In the case of non-ergodic point processes, λ may be a random variable.

Definition 2 (Voronoi perturbation) *For a proper point pattern $\varphi \subset \mathbb{R}^d$, the* Voronoi perturbation *is*

$$\mathcal{V}(\varphi) \triangleq \{x \in \varphi \colon U(V_x)\}.$$

The Voronoi perturbation is obtained by the independent random displacement of each point within its Voronoi cell. It is itself proper. The point pattern φ may be chosen as a realization of a stationary point process, or as a point process itself, in which case we denote it as Φ.

Remark. Stationarity of a point process is not required for the Voronoi cells to be finite a.s.; it suffices that the convex hull of all B with intensity measure $\Lambda(B) > 0$ equals $\mathbb{R}^d$.

Definition 3 (Stable point process) *A proper point process $\Phi \subset \mathbb{R}^d$ is* stable *if $\mathcal{V}(\Phi) \overset{\mathrm{d}}{=} \Phi$.*

It is not obvious that stable point processes exist. If they do, they can be constructed by applying the Voronoi perturbation repeatedly to any proper point pattern.

Definition 4 (Voronoi iterated point process sequence) *Let $\Phi_0 \subset \mathbb{R}^d$ be proper and define*

$$\Phi_{n+1} \triangleq \mathcal{V}(\Phi_n), \quad n = 0, 1, \ldots .$$

The sequence $(\Phi_0, \Phi_1, \Phi_2, \ldots)$ is called Voronoi iterated point process sequence *(VIPS).*

By this definition, if the VIPS converges in distribution, it converges to a stable point process. Formally, this means that there exists a probability measure P_∞ such that

$$\mathbb{P}(\Phi_n \in A) \to \mathsf{P}_\infty(A) \quad \text{as} \quad n \to \infty \tag{1}$$

for all events A. Intuitively, such stationary law P_∞ is unique, *i.e.*, it does not depend on Φ_0, only on the density λ.

III. Existence and Basic Properties

By construction, the VIPS $(\Phi_0, \Phi_1, \ldots)$ is a Markov process and belongs to the class of *iterated random functions* [1] on the space of point processes. The random functions are the independent displacements $\mathcal{V}_k$ in each step k. In the cases usually studied, the random functions are identically distributed, while here the support of each random displacement depends on the current state and thus is different for each point and in each step. Such a Markov chain is time-varying and irreversible, and standard results on the convergence to a stationary distribution

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such as *average contractivity* [2, Thm. 1], which would require to show that, for a suitable distance ℓ,

$$\mathbb{E}\log\left(\frac{\ell(\mathcal{V}(\varphi_1),\mathcal{V}(\varphi_2))}{\ell(\varphi_1,\varphi_2)}\right)<0, \tag{2}$$

are not applicable. Hence a formal proof of the existence of stable processes is elusive but is stipulated by a conjecture for which ample evidence will be provided.

Conjecture 1 *Starting with a proper point pattern or process Φ_0, the limit*

$$\Phi_\infty \triangleq \lim_{n\to\infty}\Phi_n$$

of a VIPS has a unique stationary distribution and forms a stationary point process, defined only by the density λ of the starting set Φ_0 (and all intermediate processes Φ_n). Consequently, Φ_∞ is a stable point process.

By construction, the dynamics of the VIPS exhibits the following properties:

1) Repulsion. If two points are very close, they are likely to be repelled from each other in the next iteration, since most of the Voronoi cell lies on the other side of the close neighbor.
2) Attraction. If a point lies near the centroid of its Voronoi cell (*i.e.*, it has several neighbors at a comparable distance), it is likely to be significantly closer to the nearest neighbor than the second-nearest one in the next iteration[1].

The combined forces of repulsion and attraction suggest that the VIPS tends to a point process with a certain level of regularity, rather than towards extreme clustering or a lattice (extreme repulsion). Indeed, it turns out that the stable process is a soft-core process, *i.e.*, the second-order density is smaller than λ^2 at short distances.

Proposition 1 *The stable point process is a soft-core process with pair correlation function (pcf) $g(r)=\Theta(r)$, $r\to 0$, in all dimensions.*

Proof: For a point x to have a neighbor within distance r, x needs to be within r of its nearest Voronoi boundary, and the point y in that neighboring cell needs to fall in the (hyper)ball segment $b(x,r)\cap V_y$. The first event has probability $\Theta(r)$, $r\to 0$, while the second has probability $\Theta(r^d)$, $r\to 0$, since the volume of the hyperball segment grows with r^d. So for Ripley's K function [3, Def. 6.8], we have $K(r)=\Theta(r^{d+1})$, and thus $g(r)=K'(r)/(dc_d r^{d-1})=\Theta(r)$, $r\to 0$, where $c_d=|b(o,1)|$. This indicates repulsion relative to the PPP, for which $g(r)\equiv 1$, and $g(r)>0$ for $r>0$ shows that the stable point process is not a hard-core process. ∎

This property of the pair correlation function holds also for any Φ_n, $n>0$, if Φ_0 is stationary.

Next we examine the one- and two-dimensional cases.

[1]If the cell was a disk with the point at the origin, the expected distance of the displaced point from the origin would be $2/3$ of the cell radius, so the point is likely to move out of the center

IV. The One-Dimensional Case

A. Repulsion, Attraction, and Filtering

Let $\{x_k\}_{k\in\mathbb{Z}}$ be the ordered[2] points of a point pattern $\varphi\subset\mathbb{R}$ and $\{x'_k\}_{k\in\mathbb{Z}}$ the points of $\mathcal{V}(\varphi)$ such that x'_k is the new position of x_k. This way, the ordering is preserved,

Lemma 1 (Repulsion) *Letting $m_k=(x_{k-1}+x_{k+1})/2$ be the midpoint of the left and right neighbors of x_k, we have $\mathbb{E}x'_k=(x_k+m_k)/2$,* i.e.*, the new expected position of each point is the mid-point between its current position and the mid-point of its neighbors. Equivalently, denoting by $X(\cdot)$ the discrete-time Fourier transform[3] (DTFT) of φ interpreted as a signal vector, the DTFT $\bar{X}'$ of the expected new positions $(\mathbb{E}x'_k)_{k\in\mathbb{Z}}$ is given by $\bar{X}'(\omega)=\frac{1}{2}X(\omega)(1+\cos\omega)$.*

Proof: x'_k is uniform on the interval $[(x_{k-1}+x_k)/2,(x_k+x_{k+1})/2]$, hence $\mathbb{E}x'_k=x_k/2+(x_{k-1}+x_{k+1})/4=x_k/2+m_k/2$. In the frequency domain, this relationship corresponds to a multiplication with the low-pass (LP) frequency response $(1+\cos\omega)/2$. ∎

In words, each point is expected to move half-way towards the center of its left and right neighbors. This shows that points are, on average, repelled from their closer neighbor and attracted by the further one.

Letting c_k be the length of the Voronoi interval of nucleus x_k, we have $c_k=(x_{k+1}-x_{k-1})/2=x_{k+1}-m_k$, *i.e.*, c_k does not depend on x_k itself but only on its neighbors.

The evolution can also be expressed as a move to the expected position (LP filtering) plus the addition of a white noise process of uniform random variables whose support depends on the two neighbors, *i.e.*,

$$x'_k=(x_k+m_k)/2+U([-c_k/2,c_k/2]). \tag{3}$$

As such, it constitutes a general form of a moving-average process. The first term makes the point process more regular, while the second one ensures that it does not approach a lattice. Put differently, it leads to some level of attraction since there is a probability that two points get arbitrarily close to each other. This is expressed quantitatively in the next lemma.

Lemma 2 (Attraction) *For a pattern with points $\pm 2,\pm 1,0$ in φ, with $x_0=0$, let $D=\min\{-x'_{-1},x'_1\}$ be the distance of x'_0 to its nearest neighbor. The support of D includes 0, and $\mathbb{E}D=5/6$.*

Proof: A standard calculation reveals that the distance to the new nearest neighbor has the cumulative distribution function (cdf)

$$F_1(x)=\begin{cases}-\frac{1}{3}x^3+x^2 & 0\le x<1\\ \frac{1}{3}x^3-2x^2+4x-\frac{5}{3} & 1\le x\le 2,\end{cases}$$

which has a mean of $5/6$. ∎

This shows that a point right at the center of its neighbors is likely to be closer to one of them after displacement. The second-nearest point is at mean distance $7/6$.

[2]Such that x_0 is the first non-negative point

[3]Since φ does not have a DTFT, suitable windowing is presumed.

B. One Caged Point

To make concrete statements about the evolution and the stationary distribution, we consider the single point $x_0 \in (-1/2, 1/2)$, with its neighbors fixed at $\pm 1/2$. Its Voronoi cell (interval) is $[x_0/2-1/4, x_0/2+1/4]$. In this case, we can numerically evaluate (2) for the standard Euclidean distance and obtain approximately -0.35, which shows average contractivity and, in turn, the existence of a stationary distribution.

The probability density function (pdf) after n iterations is denoted by f_n, with f_0 the pdf of x_0, and $f \equiv f_\infty$ is the stationary distribution. We also denote by $f^{(n)}$ the n-th derivative of $f\colon [0,1] \mapsto \mathbb{R}^+$ and use $f' \equiv f^{(1)}$.

The following lemma characterizes f_n and its derivatives.

Lemma 3 (Properties of the density function)

(a) For any f_0,
- $f_n(0) = 2$, $n > 0$.
- $f_n(\pm 1/2) = 0$ *and* $f'_n(-1/4) = 8$, $n > 1$.
- *Generally,* $f_n^{(m)}(\pm 1/2) = 0$, $n > m+1$.

(b) If f_0 is the uniform distribution,

$$f_n(x) = \frac{2^{(n^2+3n)/2}}{n!}(x+1/2)^n, \quad x \le -1/2+2^{-n}. \tag{4}$$

In particular, $f_n(-1/2+2^{-n}) = 2^{(3n-n^2)/2}/n!$. *For* $n = 0,1,2,3,4$, *this is* $1, 2, 1, 1/6, 1/96$.

(c) The stationary distribution f satisfies
- $f^{(n)}(\pm 1/2) = 0$ *and* $f^{(n)}(0) = 0$, $n > 0$.
- $f(\pm 1/4) = 1$, $f'(-1/4) = 8$, $f'(-3/8) = 4$.
- $f^{(n)}(-1/2 + 2^{-n-1}) = 2^{(n^2+3n+2)/2}$. *For* $n = 0,1,2,3$, *this is* $2, 8, 64, 1024$.

Proof: In each iteration, the point $x \in (-1/2, 1/2)$ is displaced according to the kernel $k(x,y) = 2\mathbf{1}_{[x/2-1/4,x/2+1/4]}(y)$, thus the distribution evolves as

$$\begin{aligned} f_{n+1}(y) &= 2\int_{-1/2}^{1/2} \mathbf{1}_{[x/2-1/4,x/2+1/4]}(y) f_n(x)\mathrm{d}x \\ &= 2\int_{\max\{-1/2,2y-1/2\}}^{\min\{1/2,2y+1/2\}} f_n(x)\mathrm{d}x. \end{aligned} \tag{5}$$

In differential form,

$$f'_{n+1}(y) = 4 f_n(2y+1/2), \quad y \le 0. \tag{6}$$

(a) Since f_n is a pdf supported on $[-1/2, 1/2]$, $f_{n+1}(0) = 2\int_{-1/2}^{1/2} f_n(x)\mathrm{d}x = 2$. Hence $f_n(0) = 2$ for any $n > 0$, irrespective of f_0. Since f_1 cannot have a discrete part (Dirac pulse), by (5), $f_2(\pm 1/2) = 0$. The other properties follow from (6).

(b) Starting with a uniform f_0, $f_1(y) = 2 - 4|y|$. The next iteration yields $f_2(y) = 16(y+1/2)^2$ for $y \le -1/4$ and $f_2(y) = 2 - 16y^2$ for $|y| \le 1/4$. Repeated application of (5) yields the result.

(c) Since the stationary distribution is unique it is given by the density for which $f_n \equiv f_{n+1}$ and thus, by (6), $f'(y) = 4f(2y+1/2)$, $y \le 0$. Repeated application of (6) shows that f is infinitely flat at $\pm 1/2$ and 0. and yields the specific values of f and $f^{(n)}$. ∎

A good approximation of the limiting distribution is

$$\tilde{f}(x) = \begin{cases} 2 - e^{1-1/(16x^2)}, & |x| \le 1/4, \\ e^{1-1/(16(1/2-|x|)^2)} & |x| \in [1/4, 1/2]. \end{cases} \tag{7}$$

The Wasserstein-1 distance between the exact distribution and this approximation is excellent[4] since the distance is less than 0.005.

There is an alternative way to show convergence and express the stationary distribution. To this end, we introduce the $\mathcal{H}$ distribution.

Definition 5 ($\mathcal{H}$ distribution) *For $n \in \mathbb{N}$, The $\mathcal{H}_n$ distribution is the distribution of the sum of independent uniform random variables*

$$H_n \triangleq \sum_{k=1}^{n} U([-2^{-k-1}, 2^{-k-1}]), \tag{8}$$

and we set $\mathcal{H}$ to be the distribution of $\lim_{n\to\infty} H_n$.

Lemma 4 *In the one-dimensional case with fixed points at $\pm 1/2$, the stationary distribution is the $\mathcal{H}$ distribution. If $x_0 = 0$, f_n is the pdf of $\mathcal{H}_n$ for all n.*

Proof: Let x_n be the interior point after n iterations with $x_0 \in (-1/2, 1/2)$ chosen arbitrarily. According to (3),

$$x_n = x_{n-1}/2 + U([-1/4, 1/4]), \quad n > 0,$$

As $n \to \infty$, the value of x_0 becomes irrelevant, hence the limiting distribution is $\mathcal{H}$. For $x_0 = 0$, $x_n \overset{\mathrm{d}}{=} H_n$. ∎

For general distributions f_0 of x_0, we have $f_n = 2^n f_0(2^n \cdot) * f_{H_n}$, where $kf_0(k\,\cdot)$ is the pdf of x_0/k and f_{H_n} is that of H_n. This is alternative representation of the limiting distribution.

Fig. 1 shows the density functions for $\mathcal{H}_n$ for $n = 1,2,3,4$. For $n = 3$ and $n = 4$, the are almost indistinguishable, so $\mathcal{H}_4$ is already very close to $\mathcal{H}$. Even $\mathcal{H}_2$ is already an excellent match. Its pdf is the piecewise linear function

$$f_{H_2}(x) = \begin{cases} 0 & |x| > \frac{3}{8} \\ 8(\frac{3}{8} - |x|) & \frac{1}{8} \le |x| < \frac{3}{8} \\ 2 & |x| \le \frac{1}{8}, \end{cases} \tag{9}$$

This is the trapezoidal function in Fig. 1.

The fact that f is infinitely flat at $\pm 1/2$ shows the repulsive nature of the process.

C. Convergence in the General Case

Here we numerically explore how quickly the statistics of the processes in the VIPS converge to the those of the stable distribution.

We focus on the interpoint intervals $d_k = x_k - x_{k-1}$. If Φ_0 is a PPP with intensity 1, $(d_k)_{k\in\mathbb{Z}}$ are iid exponential with mean 1, and their variance-to-mean ratio is 1. Conversely, if $\Phi_0 = \mathbb{Z}$, the variance is 0. In Fig. 2, the convergence of the ratio is shown for these two cases. Convergence is fast in the

[4] The distance that qualifies as an "excellent" match is quantified in https://stogblog.net/2022/06/15/how-well-do-distributions-match-a-case-for-the-mh-distance/.

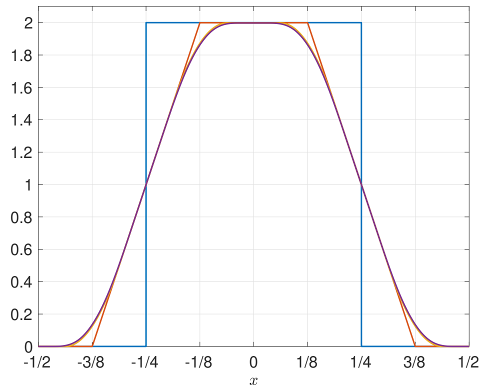


Fig. 1: Density functions of $\mathcal{H}_n$, $n \in \{1, 2, 3, 4\}$.

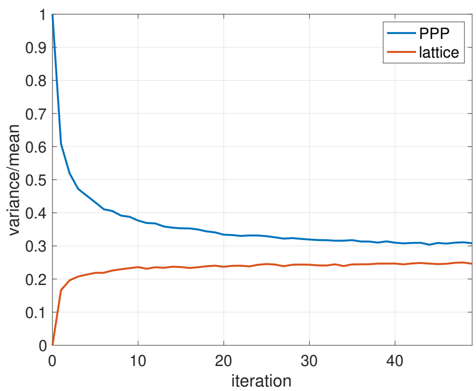


Fig. 2: Convergence of the variance-to-mean ratio in the one-dimensional case when the initial point process is a PPP and an integer lattice, respectively.

beginning, it takes only 3 iterations for 75% convergence to the limiting value of 0.26 and then slows down considerably. If the stationary intervals are approximated using a gamma variable, it corresponds closely to that of Nakagami-4 fading. Hence the difference between the PPP and the stable process is akin to that between Rayleigh fading and Nakagami-4 fading.

V. The Two-Dimensional Case

A. Convergence

To examine how quickly the VIPS converges to the stable process, we consider the mean interference-to-signal ratio (MISR) [4] at exponent 4, defined as

$$\mathsf{MISR}_4 \triangleq \mathbb{E}\left(\|x_0\|^4 \sum_{i=1}^{\infty} \|x_i\|^{-4}\right), \tag{10}$$

where the points $\{x_0, x_1, \ldots\}$ are ordered according to their distance from the origin. The MISR is a natural and simple statistic that applied in any dimension. For the PPP, $\mathsf{MISR}_4 = 1$, while for more regular point processes, $\mathsf{MISR}_4 < 1$. For the square lattice, $\mathsf{MISR}_4 \approx 1/2$ [5].

Fig. 3 shows that MISR_4 converges to its final value of about $3/4$ after only 4 iterations. So in terms of MISR_4, the SPP is precisely in between the lattice and the PPP, and the convergence speed from both is very similar. The simulation is based on 40000 points on $[-100, 100]^2$.

Compared with the one-dimensional case, the convergence is significantly faster, for two reasons: More probability mass

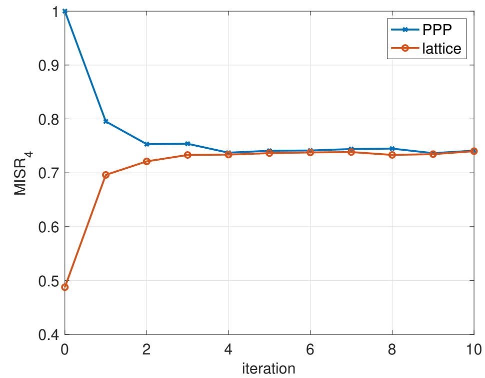


Fig. 3: Simulated convergence of MISR_4 of the two-dimensional SPP when the initial point process is a PPP and a square lattice, respectively.

of the uniform distribution on the Voronoi cells is concentrated near the boundary, and repulsion and attraction act in many directions at once.

B. J Function and Comparison with Ginibre Point Process

Letting G denote the cdf of the nearest-neighbor distance and F the empty space function (cdf of the distance from an arbitrary location to the nearest point), a repulsive process has the property that $G(r) < F(r)$ for $r > 0$ since the nearest-neighbor distance stochastically dominates the nearest-point distance. The ratio of the complementary cdfs $J(r) \triangleq (1 - G(r))/(1 - F(r))$ is called the J function [3, Def. 2.40].

It is natural to compare the SPP with the Ginibre point process (GPP) and the β-GPP [6], which are arguably the most popular soft-core models used for wireless networks. In the β-GPP, the points of a GPP are independently deleted with probability $1 - \beta$. Their pcf is $g(r) = \Theta(r^2)$, $r \to 0$, so, by Prop. 1, the GPP is more repulsive at least for small r.

For the standard GPP, $J(r) = e^{\pi r^2}$. For the β-GPP, $J(r) \sim 1 + \pi r^2$, $r \to 0$, and $\lim_{r\to\infty} J(r) = 1/(1-\beta)$ for $\beta < 1$.

Fig. 4 displays the J functions of the SPP in comparison with the GPP and 2/3-GPP. $\beta = 2/3$ is chosen so that the MISR, given by $\mathsf{MISR}_4 \approx (1 + \beta/2)^{-1}$ [7], corresponds to $\mathsf{MISR}_4 \approx 3/4$ of the SPP. We observe that despite having similar MISRs, the J function of the $2/3$-GPP plateaus at 3, while the J function of the SPP reaches at least 7 and may keep increasing way beyond. So the SPP is more regular at larger distances.

While the GPP is more tractable than the SPP, it has two key disadvantages: it is defined in two dimensions only, and it is hard to simulate [8]. Another advantage of the SPP is that it emerges naturally in several important applications in wireless networks.

VI. Applications

A. Vehicular Networks and Robot and UAV Swarms

In moderately high traffic, the positions of vehicles on a street may be well modeled by a one-dimensional SPP. Cars maintain a safe distance from each other, where the interpretation of "safe" varies from driver to driver. Impatient drivers may get quite close to another vehicle, while others

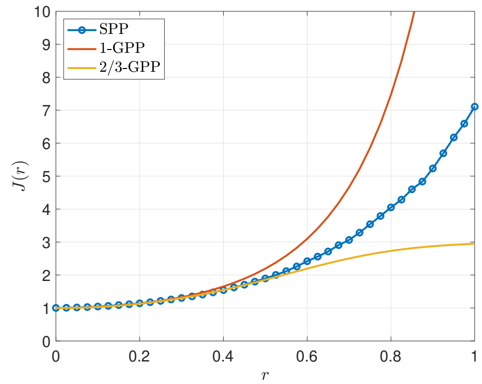


Fig. 4: J functions of the SPP (simulated), the GPP, and the 2/3-GPP of density 1.

prefer a larger distance, which leads to a variability in the inter-vehicle spacing akin to that of an SPP.

Similarly, swarms of robots or UAVs naturally attempt to keep a certain distance from each other, but the unavoidable variability is captured by the SPP in two and three dimensions, respectively.

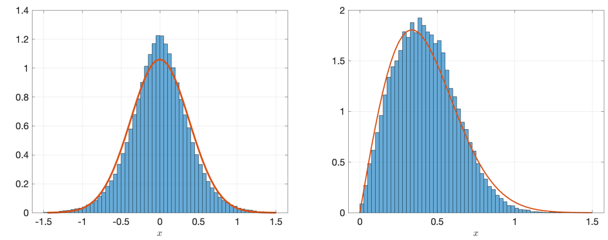
B. Base Stations and Users

Perturbation of BS locations. If an operator decides on desired base station locations φ, geographic constraints and complaints by residents may force it to find other locations with comparable coverage, without straying too far from the original locations. The new placement is thus well modeled by $\mathcal{V}(\varphi)$. After a second objection, the BSs will essentially form a stable process. For the SPP, the distribution of the signal-to-interference ratio with nearest-BS association (SIR) is related to that of the PPP by (10) as $\mathbb{P}(\mathsf{SIR} > \theta) \approx \mathbb{P}(\mathsf{SIR}_{\mathrm{PPP}} > (3/4)\theta)$ [5], *i.e.*, its regularity results in a gain of about $5/4$ dB over the PPP.

User point process. If each BS of a BS process Φ serves a user uniformly positioned in its cell, the point process of (served) users is $\mathcal{V}(\Phi)$, which is the user point process of type I defined in [9]. The fast convergence from both the PPP and the square lattice shows that the user point processes pertaining to both types of base station deployments are fairly similar, *i.e.*, the interference in the uplink is comparable. Hence the SPP is a general user model that represents user positions with good accuracy irrespective of the BS geometry.

C. Mobility Modeling

The displacements from Φ to $\mathcal{V}(\Phi)$ can naturally be interpreted as a mobility model, where the points move uniformly at random within a polygon but keep at least half the distance to any other point's current position. The movements constitute a random walk with conditionally independent displacements. A histogram of the displacements in the one-dimensional case with density 1 is shown in Fig. 5a, together with a normal approximation with variance e^{-2}. For an individual point, the displacements at times k and $k+1$ are negatively correlated, with a linear correlation coefficient -0.27. Similarly, the displacements of neighboring points are negatively correlated with $\rho = -0.17$.



(a) Empirical pdf of displacement vector in one dimension and normal approximation with variance e^{-2}.

(b) Empirical pdf of displacement distance in two dimensions and Rayleigh approximation with parameter $\sqrt{2}$ with mean $2^{-5/4} \approx 0.42$.

In two dimensions, the displacement are isotropic and close to circularly Gaussian. This is verified by comparing the empirical pdf of the lengths of the displacements with that of a Rayleigh distribution in Fig. 5b.

An interesting question is whether in a VIPS, points stay largely confined to a certain disk or drift away arbitrarily far.

VII. Conclusions

The uniform displacement of points in their Voronoi cells is a natural operation on point processes and eventually results in the *stable point process*, with rapid convergence in two and higher dimensions. The "inevitability of the Poisson process" notwithstanding [10], it is a soft-core process in all dimensions.

The results for the caged point demonstrate exactly how the distribution converges to the stationary one, which is the newly introduced $\mathcal{H}$ distribution. 86% of its probability mass is concentrated in the inner half of the unit interval, which demonstrates the repulsion of the Voronoi displacement.

It will be interesting to explore other types of displacements, for instance those supported on part of the Voronoi cell only, such as the in-disk in two dimensions, and to analyze the performance of SPP-based wireless networks in detail.

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