"Category of Verma modules"

- $c=1$ for $h \neq \frac{N^2}{4}$, objects are not mapped to/from anything

  for $h = \frac{N^2}{4}$, objects organize into two sequences

$$V_{1,0} \leftarrow V_{1,1} \leftarrow V_{1,4} \leftarrow V_{1,9} \leftarrow \cdots \quad , \quad V_{1,\frac{1}{4}} \leftarrow V_{1,\frac{9}{4}} \leftarrow V_{1,\frac{25}{4}} \leftarrow \cdots$$

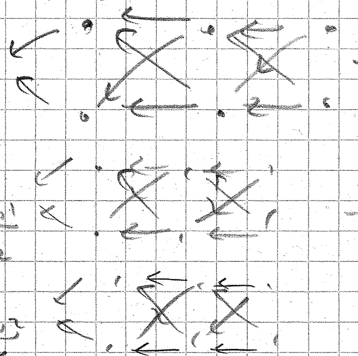
- $c = 1/2$ for $h \neq \frac{N^2-1}{48}$, objects are isolated
  (i.e. $h \neq h_{p,q}$)

  for $h = \frac{N^2-1}{48}$, objects organize into one of the following pieces (depending on N mod 24)

(1,1) $V_{1/2,\, \underbrace{h_{1,1}}_{0}}$

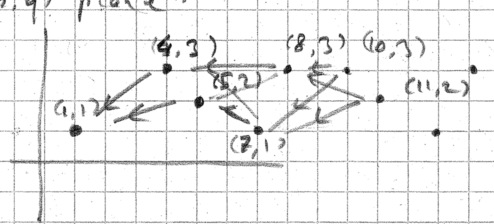
(2,1) $V_{1/2,\, \underbrace{h_{2,1}}_{1/2}}$

(1,2) $V_{1/2,\, \underbrace{h_{1,2}}_{1/16}}$



} $\underline{III}_-$ cases

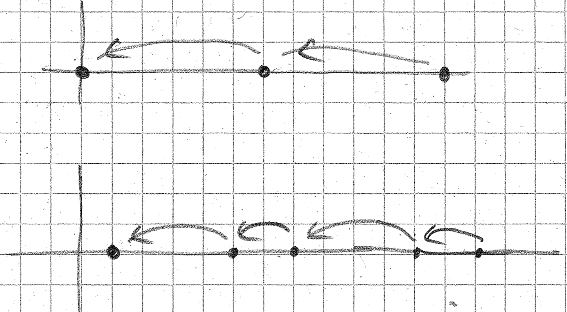
On (p,q) plane:



(0,0) $V_{1/2,\, \underbrace{h_{0,0}}_{-1/48}} \leftarrow \cdot \leftarrow \cdot \leftarrow \cdot$

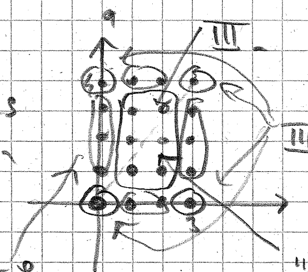
(3,0) $V_{1/2,\, \underbrace{h_{3,0}}_{143/48}} \leftarrow \cdot \leftarrow \cdot \leftarrow$

} $\underline{III}^{0,0}_-$ cases



(1,0)
(2,0)
(0,1)
(0,2)
(0,3)
} $V_{1/2,\, h_{p,q}} \leftarrow \cdot \leftarrow \cdot \leftarrow$ } $\underline{III}^0_-$ cases

- Maximal reducible modules are assoc. to (p,q) in



modulo $(p,q) \sim (3-p, 4-q)$

"Kac table"

- Category of Verma modules is organized similarly for $c = 1 - \frac{6}{m(m+1)}$ with $m \in \mathbb{Q}$,
  if $\frac{m+1}{m} = \frac{Q}{P}$, Q & P - coprime, $\in \mathbb{Z}$ then max. red. modules are $V_{c,h_{p,q}}$
  for $0 \leq p \leq P$, $0 \leq q \leq Q$ [strict inequalities - $\underline{III}$ cases]

$\rightsquigarrow$ Space of states of minimal model $\mathcal{M}(P,Q)$:

$$\mathcal{H}_{\mathcal{M}(P,Q)} = \bigoplus_{\substack{0<p<P \\ 0<q<Q \\ \frac{p}{q} > \frac{P}{Q}}} M_{c,h_{p,q}}$$

. In particular $\mathcal{M}(m, m+1)$ for $m = 3, 4, \ldots$ are <u>unitary</u>: $\mathcal{H}$ has positive definite inner product

- if $m \notin \mathbb{Q}$ then cat. of Verma modules is a collection of isolated objects for $h \neq h_{p,q}$

and diagrams $\quad V_{c,h_{p,q}} \leftarrow V_{c,h_{p,-q}}$, $\quad h_{p,-q} = h_{p,q}+pq$ $\quad$ for $p,q>0$

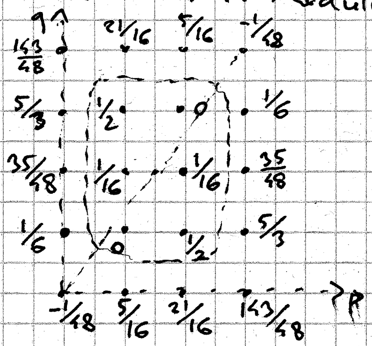
for $m \notin \mathbb{Q}$, set $\{h_{p,q}\}$ is dense in $\Big[\underbrace{\frac{c-1}{24}}_{=h_{0,0}}, +\infty\Big)$

highest weights of max. reducible Verma modules, $m \in \mathbb{Q}$ - examples

- $m=3 \Rightarrow c=\frac{1}{2}$

$$h_{p,q}=\frac{(4p-3q)^2-1}{48}$$

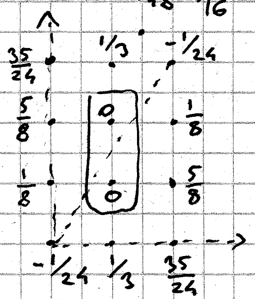
- "$\mathcal{M}(3,4)$"



- $m=2 \Rightarrow c=0$

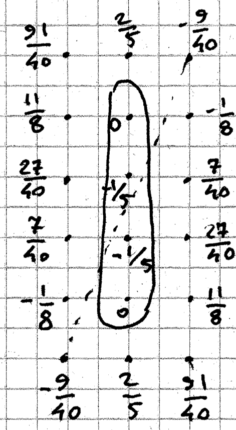
$$h_{p,q}=\frac{(3p-2q)^2-1}{24}$$

- "$\mathcal{M}(2,3)$"



- $m=\frac{2}{3} \Rightarrow c=-\frac{22}{5}$ $\quad$ - "$\mathcal{M}(2,5)$"

$$h_{p,q}=\frac{\left(\frac{5}{3}p-\frac{2}{3}q\right)^2-1}{4\cdot\frac{2}{3}\cdot\frac{5}{3}}=\frac{(5p-2q)^2-9}{40}$$



Unitarity

Virasoro irrep $M_{c,h}=V_{c,h}/\text{max. proper submodule}$ is called unitary if the inner product coming from $(L_n)^\dagger=L_{-n}$, $\langle h|h\rangle=1$ is positive-definite

$\langle h|L_1L_{-1}|h\rangle=2h \geq 0$ (if equal, then $L_{-1}|h\rangle$ is a null-vector and is modded out in $M_{c,h}$)

$\langle h|L_nL_{-n}|h\rangle=2nh+\frac{c}{12}(n^3-n) \geq 0 \quad \forall n>0$

So, $c\geq 0$, $h\geq 0$ are necessary requirements for $M_{c,h}$ to be unitary

Thm $\quad M_{c,h}$ is unitary iff

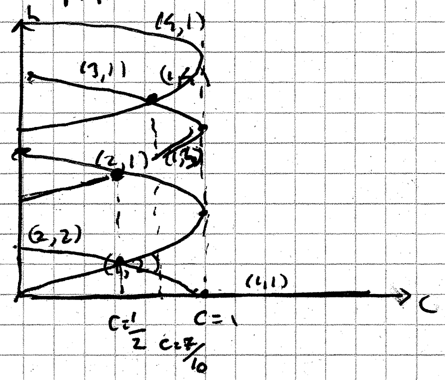
either $c\geq 1$, $h\geq 0$

or $c=1-\frac{6}{m(m+1)}$, $m=2,3,4,\dots$ and $h=h_{p,q}$ with $1\leq p\leq m-1$, $1\leq q<p$

$$h_{p,q}=\frac{((m+1)p-mq)^2-1}{4m(m+1)}$$

(Part of the) non-unitarity argument :

Draw $h_{p,q}(c) = h$ curves on (c,h)-plane:



① for $c>1$, $h \gg 0$
Gram matrices are dominated by (positive) diagonal elements
$\rightsquigarrow M_{c,h} = V_{c,h}$ are unitary

② Kac determinant does not vanish in region $c>1, h>0$
$\Rightarrow$ there $V_{c,h}$ are unitary

③ for every $c<1$, $h \neq h_{p,q}$ (for any $p,q \in \mathbb{Z}$), there exists level $N \in \mathbb{N}$ s.t. and a path connecting (c,h) to $(c>1, h>0)$ region intersecting one $h = h_{p,q}$ curve for some (p,q) s.t. $pq = N$

Thus $\det G_N(c,h) < 0$, so $M_{c,h} \stackrel{=}{} V_{c,h}$ is non-unitary

④ $c<1$, $h = h_{p,q}$ requires more careful analysis;
result is: $M_{c,h_{p,q}}$ is non-unitary unless $h_{p,q} = h_{m-p,m+1-q}$ - intersection of two $h = h_{p,q}$ curves for $m = 2,3,\dots$

Back to general story: general properties of correlators & OPE

Ward identities [reminder]:

for a rational v.f. $\varepsilon(z)\frac{\partial}{\partial z}$ with poles at $z_1 \dots z_n$ allowed, we have

- $$\delta_\varepsilon \langle \phi_1(z_1,\bar z_1) \dots \phi_n(z_n,\bar z_n) \rangle = \sum_{k=1}^{n} \langle \phi_1(z_1,\bar z_1) \dots \underbrace{\delta_\varepsilon \phi_k(z_k,\bar z_k)}_{\sum_{j=-\infty}^{\infty} \varepsilon_j^{(z_k)} \cdot L_j^{(z_k)}(\phi_k(z_k,\bar z_k))} \dots \phi_n(z_n,\bar z_n) \rangle = 0$$

where $\varepsilon_j^{(z_k)}$ are coefficients in Laurent expansion
$$\varepsilon(z_k+\zeta) = -\sum \varepsilon_j^{(z_k)} \zeta^{k+1} \frac{\partial}{\partial \zeta} ;$$
local Virasoro generators, $L_{-1}^{z_k} = \frac{\partial}{\partial z_k}$

- Similar identity for $\bar\varepsilon(\bar z)\frac{\partial}{\partial \bar z}$.

- Example: for $\varepsilon \frac{\partial}{\partial z} = \frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}$ - global conf. v.f. on $\mathbb{CP}^1$ (infinitesimal Möbius transf.) we obtain:

$\frac{\partial}{\partial z}$: $0 = \sum_{k=1}^{n} \langle \phi_1 \dots L_{-1}\phi_k \dots \phi_n \rangle = \left(\sum_{k=1}^{n} \frac{\partial}{\partial z_k}\right) \langle \phi_1 \dots \phi_n \rangle$ - translation invariance

$z\frac{\partial}{\partial z}$: $0 = \sum_{k=1}^{n} \langle \phi_1 \dots (z_k L_{-1} + L_0)\phi_k \dots \phi_n \rangle = \left(\sum_{k=1}^{n}\left(z_k \frac{\partial}{\partial z_k} + h_k\right)\right) \langle \phi_1 \dots \phi_n \rangle$ - scaling invariance

$z^2\frac{\partial}{\partial z}$: $0 = \sum_{k=1}^{n} \langle \phi_1 \dots (z_k^2 L_{-1} + 2 z_k L_0 + L_1)\phi_k \dots \phi_n \rangle \underset{\text{if } \{\phi_k\} \text{ primary}}{=\!=} \left(\sum_{k=1}^{n}\left(z_k^2 \frac{\partial}{\partial z_k} + 2 h_k z_k\right)\right) \langle \phi_1 \dots \phi_n \rangle$ - invariance under special conformal transformations

Correlators of primary fields – implications of global conf. symmetry

- $$\langle \phi(z,\bar z)\rangle = \begin{cases} \text{const}_\phi & \text{if } (h,\bar h)=0 \quad \leftarrow \text{from applying } L^z_{-1} \\ 0 & \text{otherwise} \quad \leftarrow \text{from applying } L^z_0 \end{cases}$$

- $$\langle \phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\rangle = \begin{cases} \dfrac{C_{12}}{(z_1-z_2)^{2h}(\bar z_1-\bar z_2)^{2\bar h}} & \text{if } (h_1,\bar h_1)=(h_2,\bar h_2) \\ 0 & \text{otherwise} \end{cases}$$

<u>Rem</u> operator → state : $\phi(z,\bar z) \longmapsto \lim\limits_{z,\bar z\to 0}\phi(z,\bar z)|vac\rangle =: |\phi\rangle$

operator → dual state: $\phi(z,\bar z)\longmapsto \lim\limits_{z,\bar z\to\infty} \underbrace{z^{2h}\bar z^{2\bar h}}\langle vac|\phi(z,\bar z) =: \langle\phi|$

from Jacobian of coord. trasf. $z\to z_\infty = \frac{1}{z}$

Note that then $$\langle\phi_1|\phi_2\rangle = \lim_{z,\bar z\to\infty} z^{2h_1}\bar z^{2\bar h_1}\langle\phi_1(z)\phi_2(0)\rangle = \begin{cases} C_{12} & \text{if } (h_1,\bar h_1)=(h_2,\bar h_2)\\ 0 & \text{otherwise}\end{cases}$$

<u>Rem</u> One assumes that C_{ij} is a non-deg. bilinear form on primary fields (otherwise, there would be some fields which cannot be probed by correlators).
Then one can choose a basis in primary fields s.t. $C_{ij}=\delta_{ij}$.

- $$\langle\phi_1(z_1,\bar z_1)\phi_2(z_2,\bar z_2)\phi_3(z_3,\bar z_3)\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1+h_3-h_2}(z_2-z_3)^{h_2+h_3-h_1}(\bar z_1-\bar z_2)^{\bar h_1+\bar h_2-\bar h_3}(\bar z_1-\bar z_3)^{\bar h_1+\bar h_3-\bar h_2}(\bar z_2-\bar z_3)^{\bar h_2+\bar h_3-\bar h_1}}$$

- Constants C_{ijk} not defined by $PSL_2(\mathbb{C})$-symmetry

<u>Rem</u>: $\langle\phi_1|\phi_2(z,\bar z)|\phi_3\rangle = \dfrac{C_{123}}{z^{h_2+h_3-h_1}\,\bar z^{\bar h_2+\bar h_3-\bar h_1}}$, $\langle\phi_1|\phi_2(1)|\phi_3\rangle = C_{123}$

- $$\langle\phi_1\phi_2\phi_3\phi_4\rangle = \prod_{1\le i<j\le 4}(z_i-z_j)^{-h_i-h_j+\frac{h_\Sigma}{3}}(\bar z_i-\bar z_j)^{-\bar h_i-\bar h_j+\frac{\bar h_\Sigma}{3}}\cdot f(x,\bar x)$$

$x=\dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ – cross-ratio ; $f(x,\bar x)$ – real-analytic, except maybe $x=0,1,\infty$
not determined by $PSL_2(\mathbb{C})$-symmetry

<u>Claim</u>: correlators of descendants are expressed as certain differential operators acting on correlators of corresponding primaries.

Ex 1) $$\langle(L_{-j}\phi_1)\phi_2\dots\phi_n\rangle = \overbrace{\left(\sum_{k=2}^n -\frac{1}{(z_k-z_1)^{j-1}}\frac{\partial}{\partial z_k}+\frac{j-1}{(z_k-z_1)^j}h_k\right)}^{\mathcal{L}_{-j}}\langle\phi_1\dots\phi_n\rangle$$

- Ward id. for $\varepsilon\frac{\partial}{\partial z} = -(z-z_1)^{-j+1}\frac{\partial}{\partial z}$ $\quad \varepsilon(z_k+\xi)\frac{\partial}{\partial\xi} = -(z_k-z_1)^{-j+1}\frac{\partial}{\partial\xi}+(j-1)(z_k-z_1)^{-j}\,\xi\frac{\partial}{\partial\xi}+\mathcal{O}\left(\xi^2\frac{\partial}{\partial\xi}\right)$

2) $$\langle(\underbrace{L_{-j_1}\dots L_{-j_r}\phi_1}_{=:\ \phi_1^{(j_1\dots j_r)}})\phi_2\dots\phi_n\rangle = \mathcal{L}_{-j_1}\dots\mathcal{L}_{-j_r}\langle\phi_1\dots\phi_n\rangle$$

OPE between primary fields

generally: $\Phi_1(z)\,\Phi_2(w) = \sum_{p\in I}\ \sum_{\substack{1\le k_1\le k_2\le\dots\le k_r,\ r\ge 0\\ 1\le \bar k_1\le\dots\le \bar k_s,\ s\ge 0}} C_{12}^{p,\{k,\bar k\}}\ \underbrace{(z-w)^{-h_1-h_2+h_p+K}\,(\bar z-\bar w)^{-\bar h_1-\bar h_2+\bar h_p+\bar K}}_{\text{from scaling } (L_0) \text{ symmetry}}\ \Phi_p^{\{-k,-\bar k\}}(w)$

$K = \sum k_j$, $\bar K = \sum \bar k_j$

$p\in I$: set of primary fields; sum over descendants; $\Phi_p^{\{-k,-\bar k\}}$: descendant of Φ_p at w

$$C_{12}^{p,\{k,\bar k\}} = C_{12p}\,\beta_{12}^{p\{k\}}\,\beta_{12}^{p\{\bar k\}}$$

universal rational functions of c, h_1, h_2, h_p, parameterized by partitions $\{k\}$

$\beta_{12}^{p,\emptyset} = 1$ – normalization

Rem: $\Phi_p^{(-k,-\bar k)}$ can only appear in $\Phi_1\Phi_2$ OPE if Φ_p appears.

4-point correlator of primary fields

$\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = \dots\cdot f(x,\bar x)$ — x: cross-ratio of z_1,z_2,z_3,z_4

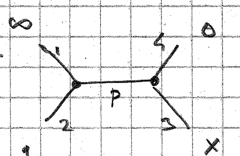
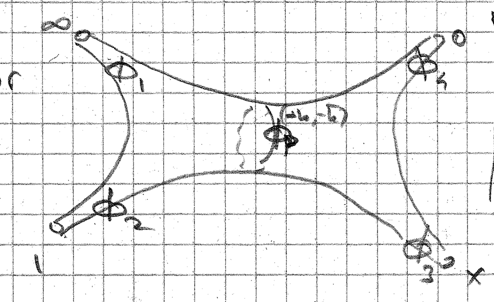
$$f(x,\bar x) = \langle\Phi_1|\,\Phi_2(1)\,\underbrace{\Phi_3(x,\bar x)\,|\Phi_4\rangle}_{\sum_p\sum_{\{k,\bar k\}} C_{34p}\,\beta_{34}^{p\{k\}}\,\beta_{34}^{p\{\bar k\}}\,x^{-h_3-h_4+h_p+K}\,\bar x^{-\bar h_3-\bar h_4+\bar h_p+\bar K}\,|\Phi_p;\{-k,-\bar k\}\rangle}$$

$|\Phi_p;\{-k,-\bar k\}\rangle$: descendant state of $|\Phi_p\rangle$

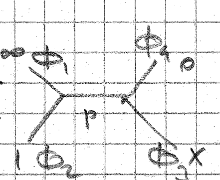
$\rightsquigarrow$ $f(x,\bar x) = \sum_{p\in I} C_{12p}\,C_{34p}\,\mathcal{F}_{34}^{21}(p|x)\,\mathcal{F}_{34}^{21}(p|\bar x)$

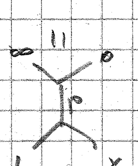
where $\mathcal{F}_{34}^{21}(p|x) = x^{-h_3-h_4+h_p}\sum_{\{k\}} x^K\,\beta_{34}^{p\{k\}}\,\gamma_{12}^{p\{k\}}$ "conformal block for 4-pt function"

where γ's come from $\dfrac{\langle\Phi_1|\Phi_2(1)|\Phi_p;\{-k,-\bar k\}\rangle}{\langle\Phi_1|\Phi_2(1)|\Phi_p\rangle} = \gamma_{12}^{p,\{k\}}\,\gamma_{12}^{p,\{\bar k\}}$

Pictures: $f(x,\bar x) = \sum_p$  or 

γ_{12}^p related to β_{12}^p via
$\gamma_{12}^{p,\{k\}} = \sum_{\{\ell\}} G^p_{\{k\},\{\ell\}}\,\beta_{12}^{p,\{\ell\}}$ — Gram matrix

Crossing symmetry: $f(x,\bar x)$ $\sum_p$ 

$\sum_p$ 

i.e. $$\boxed{\sum_p C_{12p}C_{34p}\,\mathcal{F}_{34}^{21}(p|x)\,\mathcal{F}_{34}^{21}(p,\bar x) = \sum_p C_{23p}C_{14p}\,\mathcal{F}_{32}^{14}(p|1-x)\,\mathcal{F}_{32}^{14}(p|1-\bar x)}$$

– a set of quadratic equations on C_{ijk}