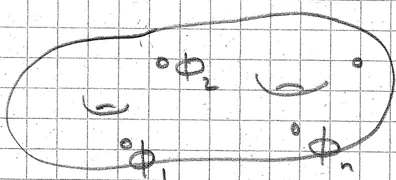
CFT, Lecture 2
(28.09.11)

CFT as a set of correlators



Surfaces with punctures (infinitesimal boundary circles), no finite boundaries

Punctures are labelled by "local quantum fields" – elements of $\mathcal{H}$ ($=$ Segal's space of states $\mathcal{H}_{S^1}$)

Partition function $Z \in (\mathcal{H}^*)^{\otimes n} \otimes \mathrm{Fun}(\mathcal{M}_{g,n})$ $\quad (*)$

Notation: $\langle \Phi_1 \ldots \Phi_n \rangle := \underset{\mathrm{Fun}(\mathcal{M}_{g,n})}{\underset{\cap}{Z(\Phi_1, \ldots, \Phi_n)}}$ – "correlator" of fields $\Phi_1, \ldots, \Phi_n$

or: $\langle \Phi_1(p_1) \ldots \Phi_n(p_n) \rangle$

punctures

<u>Rem</u> 1) There is a correction to $(*)$: Z actually depends on the metric (representing given conf. structure) and changes under Weyl tr. as

$$Z_{\Sigma, g} \xrightarrow[g \mapsto e^{2\varphi} g]{} Z_{\Sigma, g} \cdot e^{c S_{Liouville}(\varphi)}$$

where $S_{Liouville}(\varphi) = \frac{1}{2} \int_\Sigma * d\varphi \wedge d\varphi + 4 \varphi \underbrace{R_g}_{\text{Ricci scalar}} dvol$

2) One frequently considers <u>normalized</u> correlators

$$\langle \Phi_1 \ldots \Phi_n \rangle_{norm} = \frac{Z_{\Sigma_{g,n}}(\Phi_1, \ldots, \Phi_n)}{Z_{\Sigma_g}}$$ ← part. function without punctures

(so that $\langle \emptyset \rangle_{norm} = 1$)

Instead of gluing, we have the <u>operator product expansion</u>

$$\Phi_1(z) \cdot \Phi_2(w) \sim \underset{\text{runs over a basis in } \mathcal{H}}{\sum_{\Phi_3}} \underbrace{f^{\Phi_3}_{\Phi_1 \Phi_2}(z, w)}_{\substack{\text{real-analytic function} \\ \text{in a nbhd of diagonal in } \Sigma \times \Sigma}} \Phi_3(w)$$

2/2

Meaning of OPE is that such a substitution can be made inside any correlator, yielding asymptotics of $\langle \Phi_1 \cdots \Phi_n \rangle$ in nbhd of a nodal point on $\mathcal{M}_{g,n}$:

$\rightsquigarrow$ We can inductively recover n-point correlators from $(n-1)$-point correlators using (singular parts of) OPE
$\langle$similar to recovering a rational fun. from singular parts of Laurent expansions at poles$\rangle$

Conformal symmetry for correlators (Ward identities)

Recall: Witt Lie algebra = formal Laurent vector fields near $0 \in \mathbb{C}$

$$\mathrm{Witt} = \Big\{ \sum_{n \geq -N} \underset{\mathbb{C}}{\underset{\cap}{u_n}} \underbrace{\left(-z^{n+1}\frac{\partial}{\partial z}\right)}_{\ell_n} \Big\}, \quad [\ell_n, \ell_m] = (n-m)\,\ell_{n+m}$$

Also denote $\overline{\mathrm{Witt}}$ the anti-meromorphic version:

$$\overline{\mathrm{Witt}} = \Big\{ \sum_{n \geq -N} \underset{\mathbb{C}}{\underset{\cap}{\bar{u}_n}} \underbrace{\left(-\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}\right)}_{\bar{\ell}_n} \Big\}, \quad [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\,\bar{\ell}_{n+m}$$

Rem: ① Witt admits a unique central extension ($\mathrm{rk}\, H^2_{C-E}(\mathrm{Witt}) = 1$)
– Virasoro algebra: $\{1\} \to \mathbb{C} \to \mathrm{Vir} \to \mathrm{Witt} \to \{1\}$

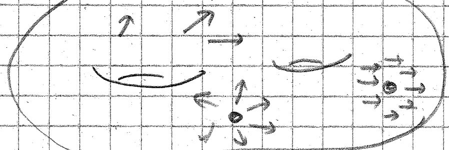
$$\mathrm{Vir} = \mathrm{Span}_{\mathbb{C}}\{L_n\}_{n \in \mathbb{Z}} \cup \{\mathbb{K}\},$$

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{n^3-n}{12}\,\delta_{n,-m}\,\mathbb{K}$$

$$[L_n, \mathbb{K}] = 0$$

③ Vir contains $\mathrm{Span}(L_{-1}, L_0, L_1) \simeq sl_2(\mathbb{C})$ as a subalgebra

② Witt and Vir are $\mathbb{Z}$-graded Lie algebras (not super-algebras!)



$$\underset{\text{conf. vector fields (complexified)}}{\mathrm{conf}_{\mathbb{C}}}(\Sigma_{g,n}) \simeq \left\{ \begin{matrix}\text{Merom. v.f.}\\ \text{on } \Sigma_{g,n}\end{matrix} \right\} \oplus \left\{ \begin{matrix}\text{Anti-merom. v.f.}\\ \text{on } \Sigma_{g,n}\end{matrix} \right\}$$

$\downarrow \pi_k$ – Laurent expansion at k^{th} puncture, $k = 1 \ldots n$

$$\Big\{ -\sum_{j \geq -N} u_j (z - z_k)^{j+1} \frac{\partial}{\partial z} - \sum_{j \geq -N} \bar{u}_j (\bar{z} - \bar{z}_k)^{j+1} \frac{\partial}{\partial \bar{z}} \Big\}$$ – local conformal algebra at z_k

$$= \mathrm{Witt}_{(z_k)} \oplus \overline{\mathrm{Witt}}_{(\bar{z}_k)}$$

Symmetry data:

$$\begin{tikzcd} \mathrm{Witt}_{(z_k)} \arrow[r, "\rho_{(z_k)}"] & \mathrm{End}(\mathcal{H}_{(z_k)}) \\ \mathrm{Vir}_{(z_k)} \arrow[u] \arrow[ur, "\tilde\rho_{(z_k)}"'] & \end{tikzcd}$$

$\rho_{(z_k)}$ – projective representation

$\tilde\rho_{(z_k)}$ – strict representation

Denote $L_n^{(z_k)} := \rho_{(z_k)}(\ell_n^{(z_k)})$

Condition: $\rho_{(z_k)}(\underbrace{\ell_{-1}}_{-\frac{\partial}{\partial z}}) \circ \Phi(z_k,\bar z_k) = \frac{\partial}{\partial z_k}\Phi(z_k,\bar z_k)$

(and $\frac{\partial}{\partial z_k}$ can be taken out of the correlator)

(Rem. I.e. $(\partial - L_{-1}, \bar\partial - \bar L_{-1})$ is a flat connection on the bundle $\begin{matrix}\mathcal{H}_{(z)}\\ \downarrow \\ z\end{matrix}$ over Σ, which allows us to compare $\mathcal{H}_{(z)}$ for different z)

Ward identity: for $v = \varepsilon(z)\frac{\partial}{\partial z} + \bar\varepsilon(\bar z)\frac{\partial}{\partial \bar z}$ a conf. v.f., we have

$$\boxed{\sum_{k=1}^n \langle \Phi_1(z_1,\bar z_1)\cdots \rho_{(z_k)}(\pi_k(v))\circ\Phi_k(z_k,\bar z_k)\cdots\Phi_n(z_n,\bar z_n)\rangle = 0}$$

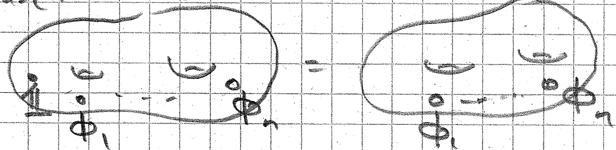
$\parallel$

$\left(\delta_{\varepsilon,\bar\varepsilon}\langle\Phi_1\cdots\Phi_n\rangle\right)$

Special fields

Identity field $\mathbb{1}$ forgets the puncture:

$\langle \mathbb{1}\,\Phi_1\cdots\Phi_n\rangle = \langle\Phi_1\cdots\Phi_n\rangle$

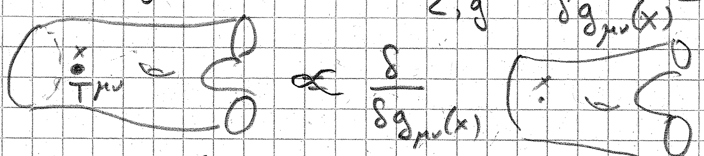


In particular, $L_{-1}\mathbb{1} = \partial_z\mathbb{1} = 0$ [we assume that there are no fields uncorrelated to everything]

OPE: $\mathbb{1}_{(z)}\Phi(w) = \Phi(w)$ (obviously)

Stress-energy tensor

- In a general Segal's QFT with Geom = {Metrics}, $T^{\mu\nu}(x)$ is a loc. quantum field defined by $\langle T^{\mu\nu}(x)\rangle_{\Sigma,g} \propto \frac{\delta}{\delta g_{\mu\nu}(x)} Z_{\Sigma,g}$



i.e. $\delta_g Z_{\Sigma,g} = \int_\Sigma \sqrt{g}\, d^Dx\, \langle T^{\mu\nu}(x)\rangle_{\Sigma,g}\, \delta g_{\mu\nu}(x)$

(cf. definition of $T^{\mu\nu}$ in class. Lagr. f.t. as $T^{\mu\nu}(x) \propto \frac{\delta S_{\Sigma,g}}{\delta g_{\mu\nu}(x)}$)

- Properties of $T^{\mu\nu}$ in class. f.t.:
  - Symmetry $T^{\mu\nu} = T^{\nu\mu}$ (by definition)
  - $\nabla_\mu T^{\mu\nu} \sim 0$ mod Euler-Lagr. equations
  - in conformal theories, $T^\mu{}_\mu = 0$
  - in 2D conf. theories, $T_{\mu\nu}$ is Weyl-invariant

- In quantum CFT: $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$,
  $\bar{\partial} T = \partial \bar{T} = 0$

  classically, $T_{z\bar{z}} = 0$ but on quantum level this is spoilt (in curved background):

  $\langle T_{z\bar{z}} \rangle \propto c \underset{\substack{\uparrow \\ \text{Ricci scalar}}}{R}$ – "trace anomaly"

- in CFT, $T(z) = L_{-2}^{(z)} \mathbb{1}^{(z)}$, $\bar{T}(\bar{z}) = \bar{L}_{-2}^{(z)} \mathbb{1}^{(z)}$

- $T, \bar{T}$ generate the action of conf. transformations on fields:

$$\delta_{\varepsilon,\bar{\varepsilon}} \Phi(w,\bar{w}) = -\frac{1}{2\pi i} \oint_{w} dz\, \varepsilon(z)\, T(z)\, \Phi(w,\bar{w}) + \frac{1}{2\pi i} \oint_{\bar{w}} d\bar{z}\, \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z}) \Phi(w,\bar{w})$$

$$\left(\rho_{(w)}(\pi_{(w)}(\varepsilon\partial + \bar{\varepsilon}\bar{\partial})) \circ \Phi(w,\bar{w}) \right)$$

Thus $\rho_{(z)}: \mathrm{Vir} \oplus \overline{\mathrm{Vir}} \to \mathrm{End}(\mathcal{H}_{(z)})$ can be recovered from $T\Phi, \bar{T}\Phi$ OPE.

Primary fields

<u>def 1</u> under conformal transf. $z \longmapsto w = w(z)$ transforms as

$$\Phi(z,\bar{z}) \longmapsto \Phi'(w,\bar{w}) = \Phi(z,\bar{z}) \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}}$$

i.e. transforms as a section of $(T_{hol}^{\otimes h} \otimes T_{anti\text{-}hol}^{\otimes \bar{h}})\Sigma$

$(h,\bar{h}) \in \mathbb{R}^2$ – "<u>conformal weight</u>" of Φ

<u>def 2</u>
$$L_n \Phi = \bar{L}_n \Phi = 0, \quad n > 0$$
$$L_0 \Phi = h \Phi, \quad \bar{L}_0 \Phi = \bar{h} \Phi$$

<u>def 3</u> (by singular part of $T\Phi$ OPE):

$$T(z)\Phi(w,\bar{w}) \sim \frac{h}{(z-w)^2} \Phi(w,\bar{w}) + \frac{1}{z-w} \partial\Phi(w,\bar{w}) + \text{reg.}$$

$$\bar{T}(\bar{z})\Phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \Phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \bar{\partial}\Phi(w,\bar{w}) + \text{reg.}$$

Another form of the Ward identity
(plugging $v = \frac{1}{w-z}\partial_w$ into general one; we assume $\phi_1 \dots \phi_n$ primary)

$$\langle T(z)\, \phi_1(z_1,\bar z_1) \dots \phi_n(z_n,\bar z_n)\rangle = \sum_{k=1}^{n} \left(\frac{h_k}{(z-z_k)^2} + \frac{1}{z-z_k}\frac{\partial}{\partial z_k} \right) \langle \phi_1(z_1,\bar z_1) \dots \phi_n(z_n,\bar z_n)\rangle$$

and likewise for $\langle \bar T \phi_1 \dots \phi_n \rangle$

Note that same identity can be recovered from $T\phi$ OPE

CFT on $\mathbb{CP}^1$ (genus = 0)

Minimal data: ① central charges of $Vir, \overline{Vir}$: $c, \bar c$;

② conf. weights $(h_i, \bar h_i)$ of all primary fields

Space of fields: $\mathcal{H} = \bigoplus_{i \in \{\text{Primaries}\}} \underbrace{V^{c,h_i} \otimes V^{\bar c, \bar h_i}}_{\text{"conf. family" of } \phi_i}$

V^{c,h_i}: Verma module for Vir with central charge c and highest weight h_i

③ Coefficients C_{ijk} in 3-point correlators of primary fields:

$$\langle \phi_i(z_1,\bar z_1)\, \phi_j(z_2,\bar z_2)\, \phi_k(z_3,\bar z_3)\rangle = C_{ijk}\, \underbrace{f_{h_i,\bar h_i; h_j,\bar h_j; h_k,\bar h_k}}_{\text{a universal function recovered from global conf. invariance}}(z_1,\bar z_1; z_2,\bar z_2; z_3,\bar z_3)$$

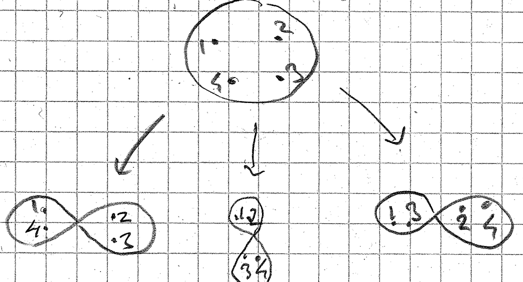
- Then one can construct all correlators:

all OPE is recovered from C_{ijk} and Ward identities,
this gives an inductive procedure allowing to construct n-point correlators,

$$\langle \phi_1 \dots \phi_n \rangle \in \left((\mathcal{H}^*)^{\otimes n} \otimes \mathrm{Fun}\big(\underbrace{(\mathbb{CP}^1 \times \overline{\mathbb{CP}^1})^{\times n}}_{\text{treat } z_k \text{ and } \bar z_k \text{ as independent complex variables}} \big) \right)^{\text{Mer. v.f.} \oplus \overline{\text{Mer. v.f.}}}$$

- "Conformal bootstrap":

there are "associativity" constraints on $\{C_{ijk}\}$ coming from existence of 4-pt function with 3 asympt. regimes:



- "Rational" CFT - with finitely many primary fields
- family with $h=\bar{h}=0$ is always present, highest vector $= \mathbb{1}$, $T = L_{-2}\mathbb{1}$ a descendant

can split holomorphic and anti-holom. dependence in corr. functions:

- $$\langle \Phi_1 \cdots \Phi_n \rangle = \sum_\alpha F_\alpha(\Phi_1, \dots, \Phi_n; z_1 \dots z_n)\, \bar{F}_\alpha(\Phi_1, \dots, \Phi_n; \bar{z}_1 \dots \bar{z}_n)$$

conformal blocks for n-point function

- a <u>finite</u> sum in case of an RCFT

- Partition function for torus:

$$Z_{torus} = \sum_{\text{primaries}} \chi_{h_i}(q)\, \chi_{\bar{h}_i}(\bar{q}) \qquad q = e^{\pi i \tau}$$

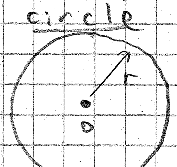
$$\overset{\shortparallel}{\mathrm{tr}_{\mathcal{H}}\, q^{L_0} \bar{q}^{\bar{L}_0}}$$

characters of Verma modules $V_{c,h_i}, V_{\bar{c},\bar{h}_i}$

= conf. blocks for torus partition function

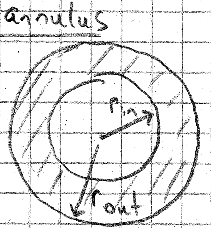
Radial quantization picture $\sim$ Segal's CFT for a subcategory of (punctured) annuli on $\mathbb{C}$

circle



$\longmapsto \mathcal{H}$

annulus

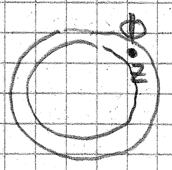


$$\longmapsto Z = \left(\frac{r_{in}}{r_{out}}\right)^{\hat{L}_0 + \hat{\bar{L}}_0} = e^{-(\tau_{out} - \tau_{in})\hat{H}}$$

where $\tau = \log r$

$\hat{H} := \hat{L}_0 + \hat{\bar{L}}_0$ - Hamiltonian

Punctured thin annulus



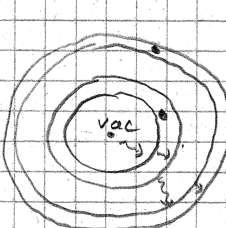
$$\underset{\mathcal{H}}{\underset{\cap}{\Phi(z)}} \longmapsto \hat{\Phi}^{\text{Schrödinger}}(z) \in \mathrm{End}(\mathcal{H})$$

$$\shortparallel$$

$Z\Big($  $\Big)$

$$\hat{\Phi}^{\text{Heisenberg}}(z) = |z|^{\hat{L}_0 + \hat{\bar{L}}_0}\, \hat{\Phi}^{\text{Schrödinger}}(z)\, |z|^{-\hat{L}_0 - \hat{\bar{L}}_0}$$

n-point correlator



$$\langle \Phi_1(z_1, \bar{z}_1) \cdots \Phi_n(z_n, \bar{z}_n) \rangle = \langle vac | \cancel{e^{-\hat{H}(\infty - \tau_n)}}\, \hat{\Phi}^S(z_n)\, e^{-\hat{H}(\tau_n - \tau_{n-1})} \cdots \hat{\Phi}^S(z_1)\, \cancel{e^{-\hat{H}(\tau_1 + \infty)}} | vac \rangle =$$

$0 < |z_1| < \dots < |z_n| < \infty$

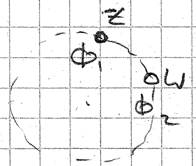
$$= \langle vac | \hat{\Phi}^{Heis}(z_n, \bar{z}_n) \cdots \hat{\Phi}^{Heis}(z_1, \bar{z}_1) | vac \rangle$$

Symmetry data: $\rho_r : Vir \oplus \overline{Vir} \longrightarrow End(\mathcal{H})$

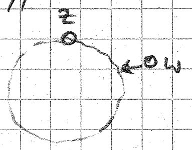
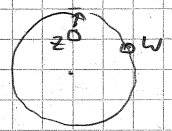
for a circle of radius r

$$L_n \longmapsto \underbrace{r^{-(\hat{L}_0+\hat{\bar{L}}_0)}\, \hat{L}_n\, r^{\hat{L}_0+\hat{\bar{L}}_0}}_{\text{Schrödinger representation}} = r^n \hat{L}_n$$

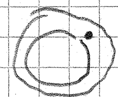
Rem Locality:

For a correlator  to be well-defined, same-time commutators should vanish $[\hat{\phi}_1(z), \hat{\phi}_2(w)] = 0$ for $|z| = |w|$, $z \neq w$

$\parallel$

$\lim$  $\lim$ 

Rem map $\mathcal{H} \longrightarrow End(\mathcal{H}) \otimes Fun(z,\bar{z})$

$\phi \longmapsto \hat{\phi}^{Heis}(z,\bar{z})$ (obtained from )

is called "state-field correspondence"

the inverse map (field-state correspondence) is:

$$\hat{\phi}(z,\bar{z}) \longmapsto \phi = \lim_{z\to 0} \hat{\phi}(z,\bar{z})\,|vac\rangle$$

OPE in radial quantization

is really an equality of operators:

$$\underset{\text{radial ordering}}{R}\, \hat{\phi}_1(z_1,\bar{z}_1)\, \hat{\phi}_2(z_2,\bar{z}_2) = \sum_{\phi_3} f^{\phi_3}_{\phi_1\phi_2}(z_1,\bar{z}_1; z_2,\bar{z}_2)\, \hat{\phi}_3(z_2,\bar{z}_2)$$

$\parallel$

$$\begin{cases} \hat{\phi}_1(z_1,\bar{z}_1) \circ \hat{\phi}_2(z_2,\bar{z}_2) & \text{if } |z_1| > |z_2| \\ \hat{\phi}_2(z_2,\bar{z}_2) \circ \hat{\phi}_1(z_1,\bar{z}_1) & \text{if } |z_1| < |z_2| \end{cases}$$