CFT Lecture 6

26.10.11

Recall • Virasoro action on $\mathcal{H}$ is constructed from stress-energy tensor:

(generally) (**) $\hat{L}_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\hat{T}(z) \in End(\mathcal{H})$ (likewise for $\hat{\bar{L}}_n$)

(from general f-la $\rho\left(\varepsilon(z)\frac{\partial}{\partial z} + \bar{\varepsilon}(\bar{z})\frac{\partial}{\partial \bar{z}}\right) = \frac{-1}{2\pi i}\oint dz\, \varepsilon(z)\hat{T}(z) + \frac{1}{2\pi i}\oint d\bar{z}\, \bar{\varepsilon}(\bar{z})\hat{\bar{T}}(\bar{z})$

and definition $\hat{L}_n := \rho\left(-z^{n+1}\frac{\partial}{\partial z}\right)$)

• OPE

(*) $$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$ (postulate of CFT)

c – central charge

implies $\left[\rho\left(\varepsilon_1(z)\frac{\partial}{\partial z}\right), \rho\left(\varepsilon_2(w)\frac{\partial}{\partial w}\right)\right] =$

$$= \frac{-1}{4\pi^2}\left(\oint dz \oint dw - \oint dz \oint dw\right) R\,\hat{T}(z)\hat{T}(w)\cdot \varepsilon_1(z)\varepsilon_2(w) =$$

Deform contours to $\oint dz \oint dw$

$$= \frac{1}{2\pi i}\oint dz\left(2\varepsilon_1(z)\partial\varepsilon_2(z)\cdot\hat{T}(z) + \varepsilon_1(z)\varepsilon_2(z)\cdot\partial\hat{T}(z) + \frac{c}{12}\varepsilon_1(z)\partial^3\varepsilon_2(z)\right)$$

$$= \rho\left([\varepsilon_1\partial, \varepsilon_2\partial]\right) + \frac{c}{12}\oint dz\, \varepsilon_1(z)\partial^3\varepsilon_2(z)$$

In particular, (*) implies Virasoro comm. relations:

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}\delta_{n,-m}(n^3-n)\,\text{id}$$

• Inverse formula for (**) is:

$$\hat{T}(z) = \sum_{n\in\mathbb{Z}} \hat{L}_n z^{-n-2}$$

• Primary fields Φ are defined as those with $T\Phi$ OPE:

$$T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial\Phi(w,\bar{w}) + \text{reg.}$$ (+ conjugate, with $\bar{h}$)

h – conf. weight

This implies the transformation property

$$\delta_\varepsilon\hat{\Phi}(w,\bar{w}) := [\rho(\varepsilon\partial), \hat{\Phi}(w,\bar{w})] = -\frac{1}{2\pi i}\oint dz\, R\,\hat{T}(z)\hat{\Phi}(w,\bar{w})\cdot\varepsilon(z) =$$

$$= -h\,\partial\varepsilon(w)\cdot\hat{\Phi}(w,\bar{w}) - \varepsilon(w)\partial\hat{\Phi}(w,\bar{w})$$

deform

Full infinitesimal transformation: $\delta_{\varepsilon\partial+\bar{\varepsilon}\bar{\partial}}\Phi(z,\bar{z}) = -\varepsilon\partial\Phi - \bar{\varepsilon}\bar{\partial}\Phi - h\partial\varepsilon\cdot\Phi - \bar{h}\bar{\partial}\bar{\varepsilon}\cdot\Phi$

Finite conf. transformation: $z \longmapsto w(z)$

$$\Phi(z,\bar{z}) \longmapsto \Phi'(w,\bar{w}) = \Phi(z,\bar{z})\left(\frac{\partial z}{\partial w}\right)^h\left(\frac{\partial\bar{z}}{\partial\bar{w}}\right)^{\bar{h}}$$ ← tensor transformation law

- local Virasoro action at puncture z_0:

$$\rho_{(z_0)}\Big(\underset{\substack{\uparrow\\ \text{may have}\\ \text{a pole at } z_0}}{\varepsilon(z)}\frac{\partial}{\partial z}\Big)\circ \underbrace{\Phi(z_0,\bar z_0)}_{\in\ \mathcal{H}_{(z_0)}\text{ - space of local fields at } z_0} := -\frac{1}{2\pi i}\oint_{z_0} dz\ \varepsilon(z)\, T(z)\, \Phi(z_0,\bar z_0)$$

Generators:
$$L_n^{(z_0)}\circ \Phi(z_0,\bar z_0) = \frac{1}{2\pi i}\oint_{z_0} dz\ (z-z_0)^{n+1}\, T(z)\, \Phi(z_0,\bar z_0)$$

Equivalently, $$T(z)\Phi(z_0,\bar z_0) = \sum_{n\in\mathbb{Z}} (z-z_0)^{-n-2}\left(L_n^{(z_0)}\circ \Phi(z_0,\bar z_0)\right)$$

- Φ is primary of weight $(h,\bar h)$ $\Leftrightarrow$
$$\begin{cases} L_n^{(z_0)}\Phi(z_0,\bar z_0)=0,\ \bar L_n^{(z_0)}\Phi(z_0,\bar z_0)=0 \quad \forall n>0\\ L_0^{(z_0)}\Phi(z_0,\bar z_0)=h\cdot\Phi(z_0,\bar z_0)\\ \bar L_0^{(z_0)}\Phi(z_0,\bar z_0)=\bar h\cdot \Phi(z_0,\bar z_0)\end{cases}$$

- <u>transformation law for T</u>

Infinitesimally TT OPE implies
$$\delta_\varepsilon T(z) = -\varepsilon\partial T - 2\partial\varepsilon\cdot T - \frac{c}{12}\partial^3\varepsilon$$

Finite version: $z\longmapsto w(z)$
$$T(z)\longmapsto T'(w) = \left(\frac{\partial w}{\partial z}\right)^{-2}\left(T(z)-\frac{c}{12}S(w,z)\right) = \left(\frac{\partial w}{\partial z}\right)^{-2}T(z)+\frac{c}{12}S(z,w)$$

$$S(w,z)=\frac{\partial_z^3 w}{\partial_z w}-\frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2$$ - Schwartzian derivative

back to free boson

<u>Recall</u>: Field $$\hat\varphi(z,\bar z) = \hat\varphi_0 - i\hat\pi_0\log(z\bar z)+\sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n}+\hat{\bar a}_n\bar z^{-n}\right)$$
$$[\hat\varphi_0,\hat\pi_0]=i\hat{1},\quad [\hat a_n,\hat a_m]=n\delta_{n,-m}\mathbb{1},\quad [\hat{\bar a}_n,\hat{\bar a}_m]=n\delta_{n,-m}\hat{\mathbb{1}}$$
$$\hat a_0 := \hat\pi_0 =: \hat{\bar a}_0$$

Space of states: $$\mathcal{H} = \text{"}\bigoplus_{\alpha\in\mathbb{R}}\text{"}\ V_{(\alpha,\alpha)}^{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}}$$
highest vector $|\alpha\rangle$ with $\hat a_0|\alpha\rangle = \hat{\bar a}_0|\alpha\rangle=\alpha|\alpha\rangle$

Virasoro generators: $$\hat L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :\hat a_m \hat a_{n-m}:$$

Stress-energy: $$\hat T(z) = -\frac{1}{2}:\partial\hat\varphi(z)\,\partial\hat\varphi(z):$$

Vertex operators (in free boson theory).

Define $\boxed{\hat V_\alpha(z,\bar z) := :e^{i\alpha\hat\varphi(z,\bar z)}:} = e^{-\alpha\sum_{n<0}\frac{\hat a_n z^{-n}+\hat{\bar a}_n \bar z^{-n}}{n}} \circ e^{-\alpha\sum_{n>0}\frac{\hat a_n z^{-n}+\hat{\bar a}_n\bar z^{-n}}{n}} \circ e^{i\alpha\hat\varphi_0} \circ e^{\alpha\hat\pi_0 \log z\bar z}$

$\alpha\in\mathbb{R}$ - "charge" of a vertex operator

Properties ① V_α is a primary field with $(h,\bar h)=\left(\frac{\alpha^2}{2},\frac{\alpha^2}{2}\right)$
(from computing $T\,V_\alpha$ OPE)

② $\lim_{z,\bar z\to 0} V_\alpha(z,\bar z)\underbrace{|vac\rangle}_{=|0\rangle} = |\alpha\rangle$

③ Correlators

$$\langle V_{\alpha_1}(z_1,\bar z_1)\,V_{\alpha_2}(z_2,\bar z_2)\rangle = \begin{cases}\frac{1}{|z_1-z_2|^{2\alpha_1^2}} & \text{if } \alpha_2=-\alpha_1\\ 0 & \text{otherwise}\end{cases}$$

$$\langle V_{\alpha_1}(z_1,\bar z_1)V_{\alpha_2}(z_2,\bar z_2)\,V_{\alpha_3}(z_3,\bar z_3)\rangle = \begin{cases}|z_1-z_2|^{2\alpha_1\alpha_2}|z_1-z_3|^{2\alpha_1\alpha_3}|z_2-z_3|^{2\alpha_2\alpha_3} & \text{if } \alpha_1+\alpha_2+\alpha_3=0\\ 0 & \text{otherwise}\end{cases}$$

$$\left\langle \prod_{k=1}^n V_{\alpha_k}(z_k,\bar z_k)\right\rangle = \begin{cases} e^{\sum_{1\le j<k\le n} 2\alpha_j\alpha_k \log|z_j-z_k|} & \text{if } \sum_k \alpha_k=0\\ 0 & \text{otherwise}\end{cases}$$

④ $V_\alpha V_\beta$ OPE:

$$R\,\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w) = |z-w|^{2\alpha\beta} :\hat V_\alpha(z,\bar z)\hat V_\beta(w,\bar w): =$$
$$= |z-w|^{2\alpha\beta}\,\hat V_{\alpha+\beta}(w,\bar w) + \text{less singular terms}$$

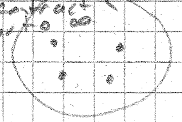
⑤ $V_\alpha(z,\bar z)\,|\beta\rangle = |z|^{2\alpha\beta}\, e^{\alpha\sum_{n>0}\frac{1}{n}(\hat a_{-n}z^n+\hat{\bar a}_{-n}\bar z^n)}|\alpha+\beta\rangle$

⑥ OPE $R\,(\underbrace{i\partial\hat\varphi(z)}_{\text{"Heisenberg current"}})\cdot\hat V_\alpha(w,\bar w) = \frac{\alpha}{z-w}\hat V_\alpha(w,\bar w) + \text{reg.}$

- Can be used to show neutrality condition for correlators:

$$0 = \oint dz\,\left\langle i\partial\varphi(z)\prod_{k=1}^n V_{\alpha_k}(w_k,\bar w_k)\right\rangle = \sum_{k=1}^n \underbrace{\oint_{w_k} dz\,\left\langle i\partial\varphi(z)\prod_{k=1}^n V_{\alpha_k}(w_k,\bar w_k)\right\rangle}_{=2\pi i\alpha_k\langle\prod_{k=1}V_{\alpha_k}\rangle \text{ due to OPE}} =$$

can contract contour to ∞



$$= 2\pi i\left\langle\prod_{k=1}^n V_{\alpha_k}\right\rangle\cdot\sum_{k=1}^n\alpha_k$$

Rem. local Heisenberg action: $a_n^{(z_0)}\circ\Phi(z_0,\bar z_0) := \frac{1}{2\pi i}\oint dz\,(z-z_0)^n\, i\partial\varphi(z)\cdot\Phi(z_0,\bar z_0)$

$\Leftrightarrow$ OPE $\quad i\partial\varphi(z)\cdot\Phi(z_0,\bar z_0) = \sum_{n\in\mathbb{Z}}\left(a_n^{(z_0)}\circ\Phi(z_0,\bar z_0)\right)(z-z_0)^{-n-1}$

Free boson with values in S^1 = "compactified" free boson

Classical picture: space of fields $\mathcal{F} = \mathrm{Maps}(\Sigma, S^1)$

cylinder $S^1 \times \mathbb{R} \simeq \mathbb{CP}^1 \setminus \{0,\infty\}$; $S^1 = \mathbb{R}/2\pi r \mathbb{Z}$, r – radius

$$\mathcal{F} = \coprod_{m\in\mathbb{Z}} \mathrm{Maps}_m(\Sigma, S^1)$$

m – winding number, i.e. $\varphi(\sigma+2\pi, \tau) = \varphi(\sigma,\tau) + 2\pi r\cdot m$

Action - same as in non-compactified case, $S = \frac{\varkappa}{2}\int_\Sigma d\tau\, d\sigma \left((\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right)$

Configuration space: $X = \coprod_{m\in\mathbb{Z}} \underbrace{\mathrm{Maps}_m(S^1,S^1)}_{X_m}$

$$\varphi(\sigma) = \varphi_0 + m r\cdot\sigma + \sum_{n\neq 0}\varphi_n e^{in\sigma} \quad \text{– in } X_m \text{ sector}$$

Hamiltonian: $H = \pi_0^2 + \underbrace{\left(\frac{mr}{2}\right)^2}_{\text{due to } \sigma\text{-dependence of 0-mode}} + \sum_{n\neq 0}\left(\pi_n\pi_{-n} + \frac{1}{4}n^2\varphi_n\varphi_{-n}\right)$

Canonical quantization: $\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_m$ – states with definite winding number

Heis. field:

$$\hat\varphi(z,\bar z) = \hat\varphi_0 + i\frac{\hat m r}{2}\log\frac{z}{\bar z} - i\hat\pi_0 \log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n \bar z^{-n}\right)$$

- Since $\hat\varphi_0$ is defined modulo $2\pi r\mathbb{Z}$, $\hat\pi_0$ has spectrum $\frac{1}{r}\mathbb{Z}$

[in Schrödinger rep. $\mathcal{H}_{z.m.} = L_2(S^1) = \{\Psi(\varphi_0)\}$, $\hat\varphi_0 = \varphi_0\cdot$, $\hat\pi_0 = -i\frac{\partial}{\partial\varphi_0}$; in basis $\Psi_e = e^{\frac{ie\varphi_0}{r}}$, $\hat\pi_0$ is diagonal with eigenvalues $\frac{e}{r}$, $e\in\mathbb{Z}$]

Introduce $\hat e = r\cdot\hat\pi_0$, $\hat e$ has spectrum $\mathbb{Z}$

then: $\hat\varphi(z,\bar z) = \hat\varphi_0 - i\frac{\hat m r}{2}\log\frac{z}{\bar z} - i\frac{\hat e}{r}\log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n \bar z^{-n}\right)$

$i\partial\hat\varphi(z) = \sum_{n\in\mathbb{Z}} \hat a_n z^{-n}$ if we set $\hat a_0 := \frac{\hat e}{r} + \frac{\hat m r}{2}$

$i\bar\partial\hat\varphi(\bar z) = \sum_{n\in\mathbb{Z}} \hat{\bar a}_n \bar z^{-n}$ if we set $\hat{\bar a}_0 := \frac{\hat e}{r} - \frac{\hat m r}{2}$

Then Hamiltonian is given by the old formula $\hat H = \hat L_0 + \hat{\bar L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\hat a_n \hat a_{-n}: + :\hat{\bar a}_n\hat{\bar a}_{-n}:$

and total momentum: $\hat P = \hat L_0 - \hat{\bar L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\hat a_n\hat a_{-n}: - :\hat{\bar a}_n \hat{\bar a}_{-n}:$

Space of states: $\mathcal{H} = \bigoplus_{(m,e)\in\mathbb{Z}^2} V^{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}}_{\left(\frac{e}{r}+\frac{mr}{2},\ \frac{e}{r}-\frac{mr}{2}\right)}$

eigenvalues for $(\hat a_0, \hat{\bar a}_0)$

For the highest vector $|e,m\rangle$,

$$\hat H\,|e,m\rangle = \frac{1}{2}\left(\left(\frac{e}{r}+\frac{mr}{2}\right)^2 + \left(\frac{e}{r}-\frac{mr}{2}\right)^2\right)|e,m\rangle = \left(\left(\frac{e}{r}\right)^2 + \left(\frac{mr}{2}\right)^2\right)|e,m\rangle$$

$$\hat P\,|e,m\rangle = \frac{1}{2}\left(\left(\frac{e}{r}+\frac{mr}{2}\right)^2 - \left(\frac{e}{r}-\frac{mr}{2}\right)^2\right)|e,m\rangle = em\,|e,m\rangle$$

Vertex operators for compactified free boson

Introduce operator $\hat\mu$ s.t. $[\hat\mu,\hat m]=i$ (& commuting with everything else) then $[\hat m, e^{ik\hat\mu}] = k\, e^{ik\hat\mu}$

thus $e^{ik\hat\mu}: \mathcal{H}_{e,m} \longrightarrow \mathcal{H}_{e,m+k}$ increases m by $+k$

likewise $[\hat\varphi_0, \underbrace{\hat\pi_0}_{``\frac{\hat e}{r}"}] = i \Rightarrow e^{ik\frac{\hat\varphi_0}{r}}: \mathcal{H}_{e,m}\to\mathcal{H}_{e+k,m}$ increases e by $+k$

Define "chiral parts" of $\hat\varphi$:

$$\hat\chi(z) := \frac12\hat\varphi_0 + \frac{\hat\mu}{r} - i\left(\frac{\hat e}{r}+\frac{\hat m r}{2}\right)\log z + \sum_{n\neq0}\frac{i}{n}\hat a_n z^{-n}$$

$$\hat{\bar\chi}(\bar z) := \frac12\hat\varphi_0 - \frac{\hat\mu}{r} - i\left(\frac{\hat e}{r}-\frac{\hat m r}{2}\right)\log\bar z + \sum_{n\neq0}\frac{i}{n}\hat{\bar a}_n \bar z^{-n}$$

so that $\hat\varphi(z,\bar z) = \hat\chi(z)+\hat{\bar\chi}(\bar z)$

Then define vertex operator with charge (e,m) as

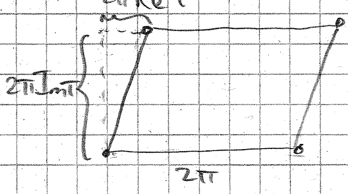
$$\boxed{\hat V_{e,m}(z,\bar z) := :e^{i\left(\frac er+\frac{mr}2\right)\hat\chi(z)}\, e^{i\left(\frac er - \frac{mr}2\right)\hat{\bar\chi}(\bar z)}:}$$

- The associated state is $\hat V_{e,m}(0)|vac\rangle = |e,m\rangle$
- $\hat V_{e,m}$ is primary with $(h,\bar h) = \left(\frac12\left(\frac er+\frac{mr}2\right)^2, \frac12\left(\frac er-\frac{mr}2\right)^2\right)$

Exercise: compute correlators of vertex operators in compactified theory

Torus partition function

Torus: $\mathbb{C}/_{2\pi\mathbb Z\oplus2\pi\tau\mathbb Z} \simeq \underbrace{\left(\mathbb C/_{2\pi\mathbb Z}\right)}_{\text{cylinder}}/_{2\pi\tau\mathbb Z}$ $\tau\in\mathbb C$, $\mathrm{Im}\,\tau>0$ - modular parameter



$$Z(\tau) = tr_{\mathcal H}\, e^{-\hat H\cdot 2\pi\, \mathrm{Im}\tau + i\hat P\cdot 2\pi\,\mathrm{Re}\tau} = tr_{\mathcal H}\, e^{-2\pi(\hat L_0+\hat{\bar L}_0)\mathrm{Im}\tau + 2\pi i(\hat L_0-\hat{\bar L}_0)\mathrm{Re}\tau} = tr_{\mathcal H}\, q^{\hat L_0}\bar q^{\hat{\bar L}_0}$$

where $q := e^{2\pi i\tau}$, $\bar q = e^{-2\pi i\bar\tau}$

Correction: Due to Schwartzian in transformation law for $T(z)$, (going from z-plane to $\zeta=\log z$ - cylinder) we have a shift $\hat H = \hat L_0+\hat{\bar L}_0 - \frac{c+\bar c}{24}$, $\hat P = \hat L_0-\hat{\bar L}_0-\frac{c-\bar c}{24}$

And the correct formula for $Z(\tau)$ is:

$$Z(\tau) = tr_{\mathcal H}\, q^{\hat L_0-\frac c{24}}\,\bar q^{\hat{\bar L}_0-\frac{\bar c}{24}}$$

[As we will see later, this shift ensures the modular symmetry $\tau\to-\tau^{-1}$ of $Z(\tau)$]

For free compactified boson, we have

$(c, \bar{c}) = (1, 1)$

$$Z(\tau) = \mathrm{tr}_{\mathcal{H}}\, q^{\hat{L}_0 - \frac{1}{24}} \bar{q}^{\hat{\bar{L}}_0 - \frac{1}{24}} = \sum_{(e,m)\in\mathbb{Z}^2} \mathrm{tr}_{\mathcal{H}_{e,m}}\, q^{\hat{L}_0 - \frac{1}{24}} \bar{q}^{\hat{\bar{L}}_0 - \frac{1}{24}} =$$

$\mathcal{H} = \bigoplus_{(e,m)\in\mathbb{Z}^2} \mathcal{H}_{e,m}$

$\mathcal{H}_{e,m}$: $\mathrm{Heis} \oplus \overline{\mathrm{Heis}}$, $\left(\frac{e}{r} + \frac{mr}{2}, \frac{e}{r} - \frac{mr}{2}\right)$

$$= \sum_{(e,m)\in\mathbb{Z}^2} q^{\frac{1}{2}\left(\frac{e}{r} + \frac{mr}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{e}{r} - \frac{mr}{2}\right)^2} \cdot (q\bar{q})^{-\frac{1}{24}} \cdot \sum_{k,l \geq 0} \underbrace{P(k)}_{\text{no. of partitions of } k} P(l)\, q^k \bar{q}^l$$

Aside

$$\sum_{k\geq 0} P(k)\, q^k \underset{|q|<1}{=} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{q^{1/24}}{\eta(\tau)}$$

where $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ – Dedekind eta-function

modular properties: $\begin{cases} \eta(\tau+1) = \eta(\tau) \cdot e^{\frac{i\pi}{12}} & \leftarrow \text{obvious} \\ \eta\left(-\frac{1}{\tau}\right) = \eta(\tau) \cdot (-i\tau)^{1/2} & \leftarrow \text{follows from Euler's pentagonal theorem} \end{cases}$

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{3j^2 - j}{2}}$$

and Poisson resummation in j

$$(*) \quad Z(\tau) = \frac{1}{\eta(\tau)\eta(\bar{\tau})} \sum_{(e,m)\in\mathbb{Z}^2} q^{\frac{1}{2}\left(\frac{e}{r} + \frac{mr}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{e}{r} - \frac{mr}{2}\right)^2}$$

Properties: ① Modular symmetry $Z(\tau+1) = Z(\tau)$

$Z\left(-\frac{1}{\tau}\right) = Z(\tau)$ – from Poisson resummation in (e, m) – relies on $q^{-\frac{1}{24}}$ shift!

② "S-duality" $Z(\underset{\uparrow}{\tau, r}) = Z\left(\tau, \frac{2}{r}\right)$

radius of target S^1

In Path Integral formalism:

$$Z = \int_{\mathrm{Maps}(\Sigma, S^1)} D\varphi \cdot e^{-S[\varphi]} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \int_{\mathrm{Maps}_{n_1,n_2}(\Sigma, S^1)} D\varphi \cdot e^{-S[\varphi]}$$

Σ: torus

(n_1, n_2): winding numbers around the two cycles

$\varphi = \varphi_0 + \varphi^{cl}_{n_1,n_2} + \tilde{\varphi}$, $\quad \varphi_0 \in \mathbb{R}/2\pi r\mathbb{Z}$, $\quad \tilde{\varphi}$: 2-periodic with $\int_\Sigma \tilde{\varphi} = 0$

$$S[\varphi] = S[\varphi^{cl}_{n_1,n_2}] + S[\tilde{\varphi}]$$

$$\varphi^{cl}_{n_1,n_2} = \sigma \cdot n_1 + t \cdot \frac{n_2 - n_1 \operatorname{Re}\tau}{\operatorname{Im}\tau}$$

$$S[\varphi^{cl}_{n_1,n_2}] = i\pi r^2 \frac{(n_2 - \tau n_1)(n_2 - \bar{\tau} n_1)}{\tau - \bar{\tau}}, \quad \text{so}$$

$$Z = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \underbrace{\underbrace{\int_{S^1} d\varphi_0}_{2\pi r}}_{\text{int. over space of solutions of EL}} \underbrace{\int_{\mathrm{Maps}(\Sigma, \mathbb{R})} D\tilde{\varphi}\, e^{-S[\tilde{\varphi}]}}_{\propto (\det'(-\Delta_\Sigma))^{-\frac{1}{2}}} \cdot e^{-S[\varphi^{cl}_{n_1,n_2}]} \propto$$

$$\propto 2\pi r \cdot \underbrace{\det{}'(-\Delta_\Sigma)^{-\frac{1}{2}}}_{\propto |\eta(\tau)|^4} \underbrace{\sum_{(n_1,n_2)\in\mathbb{Z}^2} e^{-i\pi r^2 \frac{(n_2 - \tau n_1)(n_2 - \bar{\tau} n_1)}{\tau - \bar{\tau}}}}_{}$$

$\xrightarrow{\text{Poisson resum. in } n_2}$ $(*)$ with $n_1 = m$, (dual variable to n_2) $= e$

Ref: Gawedzki, IAS lectures on CFT