02.11.11

Vertex operators for compactified free boson

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Introduce operator $\hat{\mu}$ s.t. $[\hat{\mu}, \hat{m}] = i$ then $[\hat{m}, e^{ik\hat{\mu}}] = k\, e^{ik\hat{\mu}}$
(& commuting with everything else)

thus $e^{ik\hat{\mu}} : \mathcal{H}_{e,m} \longrightarrow \mathcal{H}_{e,m+k}$ increases m by $+k$

likewise $[\hat{\varphi}_0, \underbrace{\hat{\pi}_0}_{``\frac{\hat{e}}{r}"}] = i \Rightarrow e^{i\frac{k\hat{\varphi}_0}{r}} : \mathcal{H}_{e,m} \to \mathcal{H}_{e+k,m}$ increases e by $+k$

Define "chiral parts" of $\hat{\varphi}$:

$$\hat{\chi}(z) := \frac{1}{2}\hat{\varphi}_0 + \frac{\hat{\mu}}{r} - i\left(\frac{\hat{e}}{r} + \frac{\hat{m}r}{2}\right)\log z + \sum_{n\neq 0} \frac{i}{n}\hat{a}_n z^{-n}$$

$$\hat{\bar{\chi}}(\bar{z}) := \frac{1}{2}\hat{\varphi}_0 - \frac{\hat{\mu}}{r} - i\left(\frac{\hat{e}}{r} - \frac{\hat{m}r}{2}\right)\log \bar{z} + \sum_{n\neq 0} \frac{i}{n}\hat{\bar{a}}_n \bar{z}^{-n}$$

so that $\hat{\varphi}(z,\bar{z}) = \hat{\chi}(z) + \hat{\bar{\chi}}(\bar{z})$

Then define vertex operator with charge (e,m) as

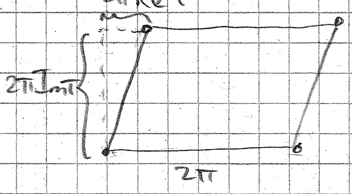
$$\boxed{\hat{V}_{e,m}(z,\bar{z}) := \, : e^{i\left(\frac{e}{r} + \frac{mr}{2}\right)\hat{\chi}(z)}\, e^{i\left(\frac{e}{r} - \frac{mr}{2}\right)\hat{\bar{\chi}}(\bar{z})} :}$$

- The associated state is $\hat{V}_{e,m}(0)|vac\rangle = |e,m\rangle$
- $\hat{V}_{e,m}$ is primary with $(h,\bar{h}) = \left(\frac{1}{2}\left(\frac{e}{r} + \frac{mr}{2}\right)^2, \frac{1}{2}\left(\frac{e}{r} - \frac{mr}{2}\right)^2\right)$

Exercise: compute correlators of vertex operators in compactified theory

Torus partition function

Torus: $\mathbb{C}/_{2\pi\mathbb{Z} \oplus 2\pi\tau\mathbb{Z}} \cong \big(\underbrace{\mathbb{C}/_{2\pi\mathbb{Z}}}_{\text{cylinder}}\big)/_{2\pi\tau\mathbb{Z}}$ $\quad \tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ - modular parameter



$$Z(\tau) = \operatorname{tr}_{\mathcal{H}} e^{-\hat{H}\cdot 2\pi \operatorname{Im}\tau + i\hat{P}\cdot 2\pi \operatorname{Re}\tau} =$$

$$= \operatorname{tr}_{\mathcal{H}} e^{-2\pi(\hat{L}_0 + \hat{\bar{L}}_0)\operatorname{Im}\tau + 2\pi i(\hat{L}_0 - \hat{\bar{L}}_0)\operatorname{Re}\tau} = \operatorname{tr}_{\mathcal{H}} q^{\hat{L}_0}\bar{q}^{\hat{\bar{L}}_0}$$

where $q = e^{2\pi i\tau}$, $\bar{q} = e^{-2\pi i\bar{\tau}}$

Correction: Due to Schwartzian in transformation law for $T(z)$, (going from z-plane to $\zeta = \log z$ - cylinder)
we have a shift $\hat{H} = \hat{L}_0 + \hat{\bar{L}}_0 - \frac{c+\bar{c}}{24}$, $\hat{P} = \hat{L}_0 - \hat{\bar{L}}_0 - \frac{c-\bar{c}}{24}$

And the correct formula for $Z(\tau)$ is:

$$Z(\tau) = \operatorname{tr}_{\mathcal{H}} q^{\hat{L}_0 - \frac{c}{24}}\bar{q}^{\hat{\bar{L}}_0 - \frac{\bar{c}}{24}}$$

[As we will see later, this shift ensures the modular symmetry $\tau \to -\tau^{-1}$ of $Z(\tau)$]

For free compactified boson, we have $(c,\bar c) = (1,1)$

$$Z(\tau) = \mathrm{tr}_{\mathcal H}\, q^{\hat L_0 - \frac{1}{24}}\, \bar q^{\hat{\bar L}_0 - \frac{1}{24}} \underset{\uparrow}{=} \sum_{(e,m)\in\mathbb Z^2} \mathrm{tr}_{\mathcal H_{e,m}}\, q^{\hat L_0 - \frac{1}{24}}\, \bar q^{\hat{\bar L}_0 - \frac{1}{24}} =$$

$\mathcal H = \bigoplus_{(e,m)\in\mathbb Z^2} \mathcal H_{e,m}$

$\mathcal H_{e,m} \leftarrow \mathrm{Heis}\oplus\overline{\mathrm{Heis}}$ $\left(\frac{e}{r}+\frac{mr}{2},\ \frac{e}{r}-\frac{mr}{2}\right)$

$$= \sum_{(e,m)\in\mathbb Z^2} q^{\frac12\left(\frac{e}{r}+\frac{mr}{2}\right)^2}\, \bar q^{\frac12\left(\frac{e}{r}-\frac{mr}{2}\right)^2} \cdot (q\bar q)^{-\frac{1}{24}} \cdot \sum_{k,\ell\geq 0} \underbrace{P(k)}_{\substack{\text{no. of partitions}\\ \text{of } k}} P(\ell)\, q^k \bar q^\ell$$

Aside

$$\sum_{k\geq 0} P(k)\, q^k \underset{|q|<1}{=} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{q^{1/24}}{\eta(\tau)}$$

where $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ - Dedekind eta-function

modular properties: $\begin{cases} \eta(\tau+1) = \eta(\tau)\cdot e^{\frac{i\pi}{12}} & \leftarrow \text{obvious} \\ \eta(-\frac{1}{\tau}) = \eta(\tau)\cdot (-i\tau)^{1/2} & \leftarrow \text{follows from Euler's pentagonal theorem} \end{cases}$

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{3j^2-j}{2}}$$

and Poisson resummation in j

$$(*)\quad Z(\tau) = \frac{1}{\eta(\tau)\eta(\bar\tau)} \sum_{(e,m)\in\mathbb Z^2} q^{\frac12\left(\frac{e}{r}+\frac{mr}{2}\right)^2}\, \bar q^{\frac12\left(\frac{e}{r}-\frac{mr}{2}\right)^2}$$

Properties: ① Modular symmetry $Z(\tau+1) = Z(\tau)$

$Z(-\frac{1}{\tau}) = Z(\tau)$ - from Poisson resummation in (e,m)

- relies on $q^{-\frac{1}{24}}$ shift!

② "S-duality" $Z(\underset{\substack{\uparrow\\ \text{radius of}\\ \text{target } S^1}}{\tau, r}) = Z(\tau, \frac{2}{r})$

In Path Integral formalism:

$$Z = \int_{\mathrm{Maps}(\Sigma, S^1)} D\varphi \cdot e^{-S[\varphi]} = \sum_{(n_1,n_2)\in\mathbb Z^2} \int_{\mathrm{Maps}_{n_1,n_2}(\Sigma, S^1)} D\varphi\, e^{-S[\varphi]}$$

Σ: torus; (n_1,n_2): winding numbers around the two cycles

$\varphi = \varphi_0 + \varphi^{cl}_{n_1,n_2} + \tilde\varphi$, $\varphi_0 \in \mathbb R/2\pi r\mathbb Z$, $\tilde\varphi$: $\mathbb Z$-periodic with $\int_\Sigma \tilde\varphi = 0$

$S[\varphi] = S[\varphi^{cl}_{n_1,n_2}] + S[\tilde\varphi]$

$\varphi^{cl}_{n_1,n_2} = \sigma\cdot n_1 + t\cdot \frac{n_2 - n_1 \mathrm{Re}\,\tau}{\mathrm{Im}\,\tau}$

$S[\varphi^{cl}_{n_1,n_2}] = i\pi r^2 \frac{(n_2-\tau n_1)(n_2-\bar\tau n_1)}{\tau-\bar\tau}$, so

$$Z = \sum_{(n_1,n_2)\in\mathbb Z^2} \underbrace{\int_{S^1_{2\pi r}} d\varphi_0}_{\text{int. over space of solutions of EL}} \underbrace{\int_{\mathrm{Maps}(\Sigma,\mathbb R)} D\tilde\varphi\, e^{-S[\tilde\varphi]}}_{\propto (\det'(-\Delta_\Sigma))^{-\frac12}} \cdot e^{-S[\varphi^{cl}_{n_1,n_2}]} \propto$$

$$\propto \boxed{2\pi r\cdot \underbrace{\det{}'(-\Delta_\Sigma)^{-\frac12}}_{\propto |\eta(\tau)|^4} \underbrace{\sum_{(n_1,n_2)\in\mathbb Z^2} e^{-i\pi r^2 \frac{(n_2-\tau n_1)(n_2-\bar\tau n_1)}{\tau-\bar\tau}}}_{\text{Poisson resum. in } n_2}} \longrightarrow (*) \text{ with } \begin{array}{l} n_1 = m \\ (\text{dual variable to } n_2) = e \end{array}$$

Ref: Gawedzki, IAS Lectures on CFT

Remark Given two CFTs, we can always construct their "product"

- Space of states $\mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II}$ - tensor product of $Vir \oplus \overline{Vir}$ - modules.
- Central charges add up: $(c, \bar{c}) = (c_I + c_{II}, \bar{c}_I + \bar{c}_{II})$
- Fields of types I and II are uncorrelated:

$$\langle \Phi_I^{(1)}(z_1,\bar{z}_1) \cdots \Phi_I^{(p)}(z_p,\bar{z}_p)\, \Phi_{II}^{(1)}(w_1,\bar{w}_1) \cdots \Phi_{II}^{(q)}(w_q,\bar{w}_q) \rangle =$$
$$= \langle \Phi_I^{(1)}(z_1,\bar{z}_1) \cdots \Phi_I^{(p)}(z_p,\bar{z}_p) \rangle_I \cdot \langle \Phi_{II}^{(1)}(w_1,\bar{w}_1) \cdots \Phi_{II}^{(q)}(w_q,\bar{w}_q) \rangle_{II}$$

- OPE $\Phi_I \Phi_{II}$ has no singular terms
- state-field correspondence. Denote $Y(\Phi; z,\bar{z}) := \hat{\Phi}(z,\bar{z})$ - field operator associated to state $|\Phi\rangle = \hat{\Phi}(0,0)|vac\rangle$.

Then for the product theory:

$$Y(\Phi_I \otimes \mathbb{1}vac_{II}; z,\bar{z}) = Y_I(\Phi_I; z,\bar{z}) \otimes \hat{\mathbb{1}}_{II}$$
$$Y(vac_I \otimes \Phi_{II}; z,\bar{z}) = \hat{\mathbb{1}}_I \otimes Y_{II}(\Phi_{II}; z,\bar{z})$$

- stress-energy tensor

$$\hat{T}(z) = \hat{T}_I(z) \otimes \hat{\mathbb{1}}_{II} + \hat{\mathbb{1}}_I \otimes \hat{T}_{II}(z)$$
$$\hat{\bar{T}}(\bar{z}) = \text{-----} \quad \text{(analogously)}$$

Example: n free (non-interacting) bosons

$$\mathcal{H} = (\mathcal{H}_{boson})^{\otimes n} \simeq \text{"}\bigoplus_{\alpha_1,\ldots,\alpha_n \in \mathbb{R}}\text{"} V_{\alpha_1 \ldots \alpha_n}^{(Heis \oplus \overline{Heis})^{\times n}}$$

$\longleftarrow$ zero-mode momenta for different species of bosons

where $\text{"}(Heis \oplus \overline{Heis})^{\times n}\text{"} = \mathrm{Span}\left(\{\hat{a}_k^{(i)}, \hat{\bar{a}}_k^{(i)}\}^{1\le i \le n}, \hat{\mathbb{1}}\right)$

with
$$[\hat{a}_k^{(i)}, \hat{a}_l^{(j)}] = k\, \delta_{k,-l}\, \delta_{ij}\, \hat{\mathbb{1}}$$
$$[\hat{\bar{a}}_k^{(i)}, \hat{\bar{a}}_l^{(j)}] = k\, \delta_{k,-l}\, \delta_{ij}\, \hat{\mathbb{1}}$$

Central charge: $(c, \bar{c}) = (n, n)$

Stress-energy tensor: $\hat{T}(z) = -\frac{1}{2} : \sum_{i=1}^{n} \partial\hat{\varphi}^{(i)}(z) \cdot \partial\hat{\varphi}^{(i)}(z) :$

$$\hat{L}_k = \frac{1}{2} \sum_{i=1}^{n} \sum_{l \in \mathbb{Z}} : \hat{a}_l^{(i)} \hat{a}_{k-l}^{(i)} :$$

Classically (on cylinder)
$$\mathcal{F} = \mathcal{F}_{boson}^{\times n} \simeq \mathrm{Maps}(\Sigma, \mathbb{R}^n)$$
$$S = \frac{1}{2\pi} \sum_{i=1}^{n} \int_\Sigma \partial_\xi \varphi^{(i)} \, \partial_{\bar{\xi}} \varphi^{(i)} \cdot \frac{i}{2} d\xi \wedge d\bar{\xi}$$

Exercise Show that $Z_{torus}\left(\begin{matrix} n \text{ compactified} \\ \text{bosons} \end{matrix}\right) = \left(Z_{torus}\left(\begin{matrix}\text{single compactified} \\ \text{boson}\end{matrix}\right)\right)^n$

[on circles of same radii]

Free fermion

On Euclidean cylinder, $S = \frac{1}{4\pi}\int_\Sigma d\tau d\sigma\, \Psi^T \gamma^0 \gamma^\mu \partial_\mu \Psi$

where γ^μ are Dirac matrices, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ $\quad \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in our case

We choose: $\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$

$\Psi = \begin{pmatrix} \psi(\tau,\sigma) \\ \bar\psi(\tau,\sigma) \end{pmatrix}$ - 2-component Majorana spinor

Space of classical fields: $\mathcal{F} = C^\infty(\Sigma) \otimes \underbrace{\mathbb{C}^{0|2}}$

parity-shifted spinor module for $Spin(2) \hookrightarrow Cliff_2 = \mathbb{C}\langle \gamma^0, \gamma^1 \rangle / \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

In complex coordinates $\xi = \tau + i\sigma$, $\bar\xi = \tau - i\sigma$:

$$S = \frac{1}{2\pi}\int_\Sigma d\tau d\sigma \left(\psi \partial_{\bar\xi}\psi + \bar\psi \partial_{\xi} \bar\psi \right)$$

$\psi, \bar\psi$ - right/left chiral (Weyl) fermions

Equations of motion: $\begin{cases} \bar\partial \psi = 0 \\ \partial \bar\psi = 0 \end{cases}$

Remark: geometrically, $\psi (d\xi)^{\otimes \frac{1}{2}}$ is a section of $K^{\otimes \frac{1}{2}}$ - square root of canonical line bundle on Σ

likewise for $\bar\psi (d\bar\xi)^{\otimes\frac{1}{2}}$

As a Hamiltonian system:

phase space $\Phi = C^\infty(S^1) \otimes \mathbb{C}^{0|2}$ $\quad$ symp. form $\omega = \frac{i}{4\pi}\oint d\sigma\, \delta\psi \wedge \delta\psi + \delta\bar\psi \wedge \delta\bar\psi$

Poisson brackets $\{\Psi^a(\sigma), \Psi^b(\sigma')\} = 2\pi i\, \delta^{ab}\, \delta(\sigma - \sigma')$

$\Psi^1 := \psi$, $\Psi^2 := \bar\psi$

Hamiltonian $H = \frac{1}{4\pi}\oint d\sigma \left(\psi \partial_\sigma \psi - \bar\psi \partial_\sigma \bar\psi \right)$

Fourier modes: $\psi(\sigma) = \sum_n e^{-in\sigma} b_n$, $\bar\psi(\sigma) = \sum_n e^{in\sigma} \bar b_n$

$\{b_n, b_m\} = i\,\delta_{n,-m}$, $\{\bar b_n, \bar b_m\} = i\,\delta_{n,-m}$

Canonical quantization:

Heisenberg field $\hat\psi(\xi) = \sum_n e^{-n\xi}\hat b_n$, $\hat{\bar\psi}(\bar\xi) = \sum_n e^{-n\bar\xi} \hat{\bar b}_n$

anti-commutation relations $[\hat b_n, \hat b_m] = \delta_{n,-m}\hat{\mathbb{1}}$

$[\hat{\bar b}_n, \hat{\bar b}_m] = \delta_{n,-m}\hat{\mathbb{1}}$

From cylinder to the plane

$\left(\frac{\partial z}{\partial \xi}\right)^{-(\frac{1}{2})} \leftarrow h_\psi$ - conf. weight of ψ

$$\xi \longmapsto z = e^{\xi} \qquad \psi_{cyl}(\xi) \longmapsto \psi_{plane}(z) = z^{-1/2}\,\psi_{cyl}(\xi)$$

$$\bar\psi_{cyl}(\bar\xi) \longmapsto \bar\psi_{plane}(\bar z) = \bar z^{-1/2}\,\bar\psi_{cyl}(\bar\xi)$$

In terms of modes: $\hat\psi(z) = \sum_n \hat b_n z^{-n-\frac{1}{2}}$

- Periodic boundary condition (P) on cylinder <=> anti-periodic on plane

$$\psi_{cyl}(\sigma+2\pi,\tau) = \psi_{cyl}(\sigma,\tau) \qquad \psi_{plane}(e^{2\pi i}\cdot z) = -\psi_{plane}(z)$$

- "Ramon sector"

Here $\boxed{\hat\psi_P(z) = \sum_{n\in\mathbb{Z}} \hat b_n z^{-n-\frac{1}{2}}}$ - summation over integer indices n

- Anti-periodic (A) condition on cylinder <=> periodic on plane

$$\psi_{cyl}(\sigma+2\pi,\tau) = -\psi_{cyl}(\sigma,\tau) \qquad \psi_{plane}(e^{2\pi i}\cdot z) = \psi_{plane}(z)$$

- "Neveu-Schwarz sector"

$\boxed{\hat\psi_A(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \hat b_n z^{-n-\frac{1}{2}}}$ - summation over half-integer n

Space of states (for a chiral fermion):

$$\mathcal{H} = \mathcal{H}_P \oplus \mathcal{H}_A \qquad \hat b_{>0}|vac_P\rangle = 0 \qquad \hat b_{>0}|vac_A\rangle = 0$$

$$\mathcal{H}_P = \mathrm{Span}\{\cdots \hat b_{-2}^{n_2}\hat b_{-1}^{n_1}\hat b_0^{n_0}|vac_P\rangle\}_{n_0,n_1,n_2,\dots\in\{0,1\}}$$

$$\mathcal{H}_A = \mathrm{Span}\{\cdots \hat b_{-5/2}^{n_{5/2}}\hat b_{-3/2}^{n_{3/2}}\hat b_{-1/2}^{n_{1/2}}|vac_A\rangle\}_{n_{1/2},n_{3/2},n_{5/2},\dots\in\{0,1\}}$$

fermionic occupation numbers (satisfy Pauli principle since $(\hat b_n)^2 = 0$, $n\neq 0$)

$\psi\psi$ propagator

P sector: $\langle\psi(z)\psi(w)\rangle_P \overset{\text{assume } |z|>|w|}{=} \langle vac_P|\hat\psi(z)\hat\psi(w)|vac_P\rangle =$

$$= \sum_{n,m\in\mathbb{Z}} \langle vac_P|\hat b_n \hat b_m|vac_P\rangle\cdot z^{-n-\frac{1}{2}}w^{-m-\frac{1}{2}} = \underbrace{\langle vac_P|\hat b_0\hat b_0|vac_P\rangle}_{1/2}\cdot z^{-\frac{1}{2}}w^{-\frac{1}{2}} + \sum_{\substack{n>0\\ n\in\mathbb{Z}}}\underbrace{\langle vac_P|\hat b_n\hat b_{-n}|vac_P\rangle}_{1} z^{-n-\frac{1}{2}}w^{n-\frac{1}{2}}$$

$$= \frac{1}{2}(zw)^{-\frac{1}{2}} + (zw)^{-\frac{1}{2}}\sum_{n=1}^{\infty}\left(\frac{w}{z}\right)^n = \boxed{\frac{1}{2}\,\frac{(z/w)^{1/2}+(w/z)^{1/2}}{z-w}}$$

A sector

$$\langle\psi(z)\psi(w)\rangle_A = \langle vac_A|\hat\psi(z)\hat\psi(w)|vac_A\rangle = \sum_{\substack{n\in\mathbb{Z}+\frac{1}{2}\\ n>0}}\langle vac_P|\hat b_n\hat b_{-n}|vac_P\rangle z^{-n-\frac{1}{2}}w^{n-\frac{1}{2}} = \frac{1}{z-w}$$

Remark
- Note that $\langle\psi\psi\rangle_P$ is not translation invariant (whereas $\langle\psi\psi\rangle_A$ is)
  This suggests that $|vac_P\rangle$ is not the true vacuum, whereas $|vac_A\rangle$ is.
- In both sectors we have $\langle\psi(z)\psi(w)\rangle = -\langle\psi(w)\psi(z)\rangle$

Stress-energy tensor

classically: $T(z) = -\frac{1}{2}\psi(z)\partial\psi(z)$, for chiral fermion $\bar{T} \equiv 0$

A sector: $\hat{T}(z) := -\frac{1}{2} :\hat{\psi}(z)\partial\hat{\psi}(z):$

TT OPE computed from $\psi\psi$ propagator, using Wick's thm and yields

$$(*) \quad R\,\hat{T}(z)\hat{T}(w) = \frac{1}{4}\frac{\hat{\mathbb{1}}}{(z-w)^4} + \frac{2\hat{T}(z)}{(z-w)^2} + \frac{\partial\hat{T}(z)}{(z-w)} + \text{reg.}$$

Hence $c = 1/2$ (and $\bar{c} = 0$ since $\bar{T}\bar{T} \equiv 0$)

P sector: Define $\hat{T}^{naive}(z) := -\frac{1}{2} :\hat{\psi}(z)\partial\hat{\psi}(z):$

Then $\hat{T}^{naive}\hat{T}^{naive}$ has wrong OPE!

The good definition is: $\hat{T}(z) := \lim_{w\to z}\left(-\frac{1}{2}\hat{\psi}(z)\partial\hat{\psi}(w) + \frac{1}{2}\frac{\hat{\mathbb{1}}}{(z-w)^2}\right)$

Then: $\hat{T}(z) = \hat{T}^{naive}(z) + \frac{1}{16z^2}$ has OPE $(*)$ also with $c = 1/2$

Shift means that $\langle \hat{T}(z)\rangle_P$ ~~naive~~ $= \frac{1}{16z^2}$ - nonzero vacuum energy in P sector

Virasoro generators

- Can be obtained from $\hat{T}(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n$

A sector: $\hat{L}_n = \sum_{m\in\mathbb{Z}+\frac{1}{2}} \left(\frac{m}{2}+\frac{1}{4}\right) :\hat{b}_{n-m}\hat{b}_m:$

P sector: $\hat{L}_n = \sum_{m\in\mathbb{Z}} \left(\frac{m}{2}+\frac{1}{4}\right) :\hat{b}_{n-m}\hat{b}_m: - \delta_{n,0}\frac{\hat{\mathbb{1}}}{16}$

In particular, $\hat{L}_0|vac_A\rangle = 0$

$\hat{L}_0|vac_P\rangle = \frac{1}{16}|vac_P\rangle$

$[\hat{L}_0, \hat{b}_n] = -n\,\hat{b}_n$ (both for A and P sectors)

A-states:

L_0 eigenvalue	states
0	$\lvert vac_A\rangle$
1/2	$b_{-1/2}\lvert vac_A\rangle$
1	- - - - (nothing)
3/2	$b_{-3/2}\lvert vac_A\rangle$
2	$b_{-3/2}b_{-1/2}\lvert vac_A\rangle$
5/2	$b_{-5/2}\lvert vac_A\rangle$
3	$b_{-5/2}b_{-1/2}\lvert vac_A\rangle$

P-states

L_0 eigenvalue	states	
1/16	$\lvert vac_P\rangle$	$b_0\lvert vac_P\rangle$
1+1/16	$b_{-1}\lvert vac_P\rangle$	$b_{-1}b_0\lvert vac_P\rangle$
2+1/16	$b_{-2}\lvert vac_P\rangle$	$b_{-2}b_0\lvert vac_P\rangle$
3+1/16	$b_{-3}\lvert vac_P\rangle$ $b_{-2}b_{-1}\lvert vac_P\rangle$ ~~$b_{-4}\lvert vac_P\rangle$, $b_{-3}b_{-1}$~~	$b_{-3}b_0\lvert vac_P\rangle$ $b_{-2}b_{-1}b_0\lvert vac_P\rangle$

States $\lvert vac_A\rangle$, $b_{-\frac{1}{2}}\lvert vac_A\rangle$, $\lvert vac_P\rangle$, $b_0\lvert vac_P\rangle$ are Virasoro-primary (i.e. killed by $\hat{L}_{>0}$)

$\lvert vac_A\rangle = \lvert 0\rangle$, $b_{-\frac{1}{2}}\lvert vac_A\rangle = \lvert 1/2\rangle$, $\lvert vac_P\rangle = \lvert 1/16\rangle_+$, $b_0\lvert vac_P\rangle = \lvert 1/16\rangle_-$

Space of states splits into 4 conformal families (highest weight Virasoro modules = some quotients of Verma modules)

$$\mathcal{H} = \underbrace{V_0^{Vir} \oplus V_{1/2}^{Vir}}_{\mathcal{H}_A} \oplus \underbrace{V_{1/16}^{Vir} \oplus V_{1/16}^{Vir}}_{\mathcal{H}_P}$$

	V_0^{Vir}	$V_{1/2}^{Vir}$	$V_{1/16}^{Vir}$	$V_{1/16}^{Vir}$
highest vector (pseudovacuum)	$\lvert 0\rangle = \lvert vac_A\rangle$	$\lvert \tfrac12\rangle = b_{-1/2}\lvert vac_A\rangle$	$\lvert \tfrac1{16}\rangle_+ = \lvert vac_P\rangle$	$\lvert \tfrac1{16}\rangle_- = b_0\lvert vac_P\rangle$
$\mathbb{Z}_2$ grading $(-1)^F$	even	odd	even	odd

Field-state correspondence

$\lvert 0\rangle \longmapsto \mathbb{1}$ ← primary $(h,\bar h) = (0,0)$

$\lvert \tfrac12\rangle \longmapsto \psi(z)$ - primary $(h,\bar h) = (\tfrac12, 0)$, odd

$\lvert \tfrac1{16}\rangle_+ \longmapsto \sigma(z)$ ← even

$\lvert \tfrac1{16}\rangle_- \longmapsto \mu(z)$ ← odd

"twist fields" - primary with $(h,\bar h) = (\tfrac1{16}, 0)$

$$\psi(z)\,\sigma(w) \sim (z-w)^{-1/2}\mu(w) + \dots$$

In particular $\lvert vac_P\rangle = \sigma(0)\lvert vac_A\rangle$, so e.g. $\langle \psi\psi\rangle_P$ that we computed is actually a 4-pt correlator $\langle \sigma(\infty)\psi(z)\psi(w)\sigma(0)\rangle$

<u>Non-chiral fermion</u> (pairing right-chiral to left-chiral)

- require P/A boundary condition is same for ψ and $\bar\psi$
- impose $\hat{b}_0 \equiv \hat{\bar{b}}_0$

Then $\mathcal{H}$ has six conformal families:

$$\mathcal{H}_{\text{non-chiral}} = \underbrace{V_{(0,0)}^{Vir\oplus\overline{Vir}} \oplus V_{(\frac12,0)}^{Vir\oplus\overline{Vir}} \oplus V_{(0,\frac12)}^{Vir\oplus\overline{Vir}} \oplus V_{(\frac12,\frac12)}^{Vir\oplus\overline{Vir}}}_{\text{A sector}} \oplus \underbrace{V_{(\frac1{16},\frac1{16})_+}^{Vir\oplus\overline{Vir}} \oplus V_{(\frac1{16},\frac1{16})_-}^{Vir\oplus\overline{Vir}}}_{\text{P sector}}$$

$(-1)^F$ parity	even	odd	odd	even	even	odd
highest vector	$\lvert vac_A\rangle$	$b_{-\frac12}\lvert vac_A\rangle$	$\bar b_{-\frac12}\lvert vac_A\rangle$	$b_{-\frac12}\bar b_{-\frac12}\lvert vac_A\rangle$	$\lvert vac_P\rangle$	$b_0\lvert vac_P\rangle$
primary field	$\mathbb{1}$	$\psi(z)$	$\bar\psi(\bar z)$	$\varepsilon \propto \psi(z)\bar\psi(\bar z)$	$\sigma(z,\bar z)$	$\mu(z,\bar z)$

<u>Remark</u> - This is a CFT equivalent to the Ising model at critical temperature