CFT Lecture 12

18.05.11

Today:
- comments on quantization
- quantization of the free scalar field on a cylinder

Classical vs. quantum mechanics

	(Ham.) Class. mech	quant. mech.
states	points of Φ (phase space)	rays in $\mathcal{H}$ (Hilb. space)
dynamics	$\mathrm{Flow}_t\{H,\cdot\}$	$\lvert\psi\rangle \longmapsto e^{-\frac{i\hat{H}t}{\hbar}}\lvert\psi\rangle = \mathcal{U}(t)\lvert\psi\rangle$ evol. operator
observables	$\mathcal{O}\in C^\infty(\Phi)$	$\hat{\mathcal{O}}\in \mathrm{Herm}(\mathcal{H})$
pairing of observables to states (measurement)	$\mathcal{O}(p)\in\mathbb{R}$ $(p\in\Phi)$	$\frac{\langle\psi\lvert\hat{\mathcal{O}}\rvert\psi\rangle}{\langle\psi\vert\psi\rangle}$ - expectation value
continuous symmetries	$I\in C^\infty(\Phi)$, $\{H,I\}=0$ $\overset{\uparrow}{\text{i. of m.}}$ $\mathrm{Flow}_\alpha\{I,\cdot\}$ - symmetry action on states	$\hat{I}\in\mathrm{Herm}(\mathcal{H})$, $[\hat{H},\hat{I}]=0$ $e^{-\frac{i\hat{I}\alpha}{\hbar}}$ - sym. action on states inf. $i\hbar\frac{d}{d\alpha}\lvert\psi\rangle_\alpha = \hat{I}\lvert\psi\rangle_\alpha$

Rem
- dynamics in QM:

Schrödinger picture

evol. acts on states

fin: $\lvert\psi\rangle_{t_1} = e^{-\frac{i\hat{H}(t_1-t_0)}{\hbar}}\lvert\psi\rangle_{t_0}$

inf: $i\hbar\frac{d}{dt}\lvert\psi\rangle_t = \hat{H}\lvert\psi\rangle_t$ (Schrödinger eq.)

but $\frac{\partial}{\partial t}\hat{\mathcal{O}} = 0$

vs.

Heisenberg picture

evol. acts on observables; states are time-independent

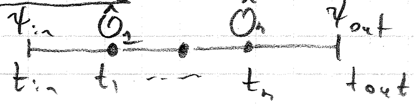
fin: $\hat{\mathcal{O}}_{t_1} = e^{i\frac{\hat{H}(t_1-t_0)}{\hbar}}\hat{\mathcal{O}}_{t_0}e^{-i\frac{\hat{H}(t_1-t_0)}{\hbar}}$

inf: $\frac{d}{dt}\hat{\mathcal{O}}_t = \frac{i}{\hbar}[\hat{H},\hat{\mathcal{O}}_t]$ Heisenberg eq.

$\frac{d}{dt}\lvert\psi\rangle = 0$

- equivalent in the sense $\langle\hat{\mathcal{O}}\rangle_{\lvert\psi\rangle_t} = \langle\hat{\mathcal{O}}_t\rangle_{\lvert\psi\rangle}$

- Correlators



$= \langle\psi_{out}\rvert\,\mathcal{U}(t_{out}-t_n)\hat{\mathcal{O}}_n\mathcal{U}(t_n-t_{n-1})\cdots\hat{\mathcal{O}}_1\mathcal{U}(t_1-t_{in})\lvert\psi_{in}\rangle$ (- Schrödinger)

$= \underbrace{\overbrace{\langle\psi_{out}\rvert}}_{\langle\psi_{out}\rvert\,\mathcal{U}(t_{out}-t_{in})}\hat{\mathcal{O}}_n(t_n)\cdots\hat{\mathcal{O}}(t_1)\lvert\psi_{in}\rangle$ (- Heisenberg)

- **VEVs** = correlators with $\lvert\psi_{in}\rangle = \lvert\psi_{out}\rangle$ = vacuum - lowest eigenstate of $\hat{H}$

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Canonical quantization

$$\Phi = T^*X \rightsquigarrow \mathcal{H} = L_2(X)$$

$$x^i \longmapsto \hat{x}^i : \psi(x) \longmapsto x^i \psi(x)$$

$$p_i \longmapsto \hat{p}_i : \psi(x) \longmapsto -i\hbar \frac{\partial}{\partial x^i} \psi(x)$$

$$\{p_i, x^j\} = \delta_i^j \rightsquigarrow [\hat{p}_i, \hat{x}^j] = -i\hbar \delta_i^j$$

$$H(x,p) \longmapsto \hat{H} = H(\hat{x}, \hat{p})$$ - ordering problem

- likewise for observables & symmetries

Class. symmetries may be <u>non-quantizable</u>, i.e. $[\hat{H}, \hat{I}] \neq 0$

or they may be quantizable but the symmetry group / Lie algebra may be deformed e.g. to a central extension

for more complicated $\Phi \rightsquigarrow$ geometric quantization

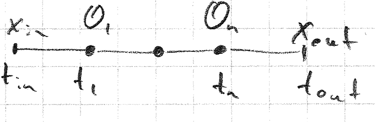
Path integral quantization

$$\Phi = T^*X \rightsquigarrow \mathcal{H} = L_2(X)$$

Evol. operator: $$\langle x_{out} | \mathcal{U}(t_{out} - t_{in}) | x_{in} \rangle = \int_{\substack{x(t_{in}) = X_{in} \\ x(t_{out}) = X_{out}}} \mathcal{D}[x(t)]_{t_{in}}^{t_{out}} \, e^{\frac{i}{\hbar} S[x(\tau)]_{t_{in}}^{t_{out}}}$$ - conf. space path integral

$$= \int_{\substack{x(t_{in}) = X_{in} \\ x(t_{out}) = X_{out}}} [\mathcal{D}x(\tau)\, \mathcal{D}p(\tau)] \, e^{\frac{i}{\hbar} \int_{t_{in}}^{t_{out}} d\tau \, (p_i \dot{x}^i - H)}$$ ← phase space path integral

Correlators:



$$= \int_{\substack{x(t_{in}) = X_{in} \\ X(t_{out}) = X_{out}}} [\mathcal{D}x(\tau)\, \mathcal{D}p(\tau)] \, e^{\frac{i}{\hbar} \int_{t_{in}}^{t_{out}} d\tau \, (p_i \dot{x}^i - H)} O_1(x(t_1), p(t_1)) \cdots \cdots O_n(x(t_n), p(t_n))$$

$$\begin{tikzcd}
\boxed{\text{Operator formalism}} \arrow[d, bend right, "\text{heat kernel asympt. of short-time evolution}"'] \\
\boxed{\text{Path-integral formalism}} \arrow[u, bend right, "\text{computing matrix elts by stationary phase or by discretization of time}"']
\end{tikzcd}$$

Rem: ordering problem does not go away in PI formalism - it becomes the problem of regularization of functional determinant / Feynman diagrams or the problem of discretization of PI.

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Scalar field on $\mathbb{R}\times S^1_{\text{discr}}$

continuum

Lag: $$S = \frac{2\pi R}{N}\sum_{k=0}^{N-1}\int dx^0\left(\frac12\dot\varphi_k^2 - \frac12\left(\frac{N}{2\pi R}\right)^2(\varphi_{k+1}-\varphi_k)^2 - \frac{m^2}{2}\varphi_k^2\right)$$

$$\{\varphi_k\in C^\infty(\mathbb{R})\}$$

$$S = \int dx^0 dx^1\left(\frac12\partial_0\varphi\,\partial_0\varphi - \frac12\partial_1\varphi\,\partial_1\varphi - \frac{m^2}{2}\varphi^2\right)$$

$$\varphi\in C^\infty(\mathbb{R}\times S^1)$$

Ham $\{\varphi_k, \pi_k = \frac{2\pi R}{N}\dot\varphi_k\}$, $\{\pi_k,\varphi_\ell\} = \delta_{k\ell}$

$$H = \sum_k \frac{N}{2\pi R}\frac{\pi_k^2}{2} + \frac12\frac{N}{2\pi R}(\varphi_{k+1}-\varphi_k)^2 + \frac{2\pi R}{N}\frac{m^2}{2}\varphi_k^2$$

$\pi(x^1) = \dot\varphi(x^1)$, $\{\pi(x^1),\varphi(x^{1\prime})\} = \delta(x^1-x^{1\prime})$

$$H = \int_0^{2\pi R} dx^1\left(\frac{\pi^2}{2} + \frac12(\partial_1\varphi)^2 + \frac{m^2}{2}\varphi^2\right)$$

Fourier modes

$$\varphi_k = \sum_{\ell=0}^{N-1} e^{2\pi i\frac{k\ell}{N}}\tilde\varphi_\ell\ ,\quad \pi_k = \sum_{\ell=0}^{N-1} e^{-2\pi i\frac{k\ell}{N}}\tilde\pi_\ell\cdot\frac1N$$

$$\tilde\varphi_k = \frac1N\sum_{\ell=0}^{N-1} e^{-2\pi i\frac{k\ell}{N}}\varphi_\ell \qquad \tilde\pi_k = \sum_{\ell=0}^{N-1} e^{+2\pi i\frac{k\ell}{N}}\pi_\ell$$

$$\{\tilde\pi_k,\tilde\varphi_\ell\} = \delta_{k,+\ell}$$

reality of φ_k,π_k → $\tilde\varphi_k = (\tilde\varphi_{-k})$, $\tilde\pi_k = (\tilde\pi_{-k})$

$$\varphi(x^1) = \sum_{\ell=-\infty}^{\infty} e^{i\frac{\ell x}{R}}\tilde\varphi_\ell \qquad \pi(x) = \sum_{\ell=-\infty}^{\infty} e^{-i\frac{\ell x}{R}}\tilde\pi_\ell\cdot\frac{1}{2\pi R}$$

$\{\tilde\pi_k,\tilde\varphi_\ell\} = \delta_{k,\ell}$

← ''

$$S = 2\pi R\int dx^0\sum_{\ell=0}^{N-1}\frac12\dot{\tilde\varphi}_\ell\dot{\tilde\varphi}_{-\ell} - \frac{\omega_\ell^2}{2}\tilde\varphi_\ell\tilde\varphi_{-\ell}\ ,\quad \boxed{\omega_\ell^2 = m^2 + \left(\frac{N}{\pi R}\sin\frac{\pi\ell}{N}\right)^2}$$

$$H = \sum_{\ell=0}^{N-1}\frac{\tilde\pi_\ell\tilde\pi_{-\ell}}{2\cdot(2\pi R)} + \frac{\omega_\ell^2}{2}(2\pi R)\cdot\tilde\varphi_\ell\tilde\varphi_{-\ell}$$

$\sum_{-\infty}^{\infty}$; $\omega_\ell^2 = m^2 + \frac{\ell^2}{R^2}$

Can. quant: $\tilde\varphi_k \longmapsto \hat{\tilde\varphi}_k$, $\tilde\pi_k\longmapsto\hat{\tilde\pi}_k$

$[\hat{\tilde\pi}_k,\hat{\tilde\varphi}_\ell] = -i\hbar\delta_{k\ell}$

$(\hat{\tilde\varphi}_k)^+ = \hat{\tilde\varphi}_{-k}$

$(\hat{\tilde\pi}_k)^+ = \hat{\tilde\pi}_{-k}$

$$\hat H = \sum_{\ell=0}^{N-1}\frac{\hat{\tilde\pi}_\ell\hat{\tilde\pi}_{-\ell}}{2\cdot(2\pi R)} + \frac{\omega_\ell^2}{2}(2\pi R)\,\hat{\tilde\varphi}_\ell\hat{\tilde\varphi}_{-\ell}$$

creation/annihil. op.

$$\hat a_k = \frac{1}{\sqrt2}\left(\sqrt{2\pi R\omega_k}\,\hat{\tilde\varphi}_k + \frac{i\hat{\tilde\pi}_k}{\sqrt{2\pi R\omega_k}}\right)$$

$$\hat a_k^+ = \frac{1}{\sqrt2}\left(\sqrt{2\pi R\omega_k}\,\hat{\tilde\varphi}_{-k} - \frac{i\hat{\tilde\pi}_{-k}}{\sqrt{2\pi R\omega_k}}\right)$$

$\cancel{(\hat a_k)^+ = \hat a^+_{-k}}$

$[\hat a_k,\hat a^+_\ell] = \delta_{k\ell}$ $\qquad \frac{a_k a_k^+ + a_k^+ a_k}{2} = \frac12\left(2\pi R\omega_k\hat{\tilde\varphi}_k^2 + \frac{\hat{\tilde\pi}_k^2}{2\pi R\omega_k}\right)$

$$\hat H = \sum_{k=0}^{N-1}\frac12(\hat a_k\hat a_k^+ + \hat a_k^+\hat a_k)\cdot\omega_k = \sum_{k=0}^{N-1}\left(\hat a_k^+\hat a_k + \frac12\right)\cdot\omega_k$$

$[H,\hat a_k^+] = +\omega_k\hat a_k^+$

$\hat H\longmapsto :\hat H: = \sum\hat a^+_k\hat a_k\cdot\omega_k$

"$\hat H_{norm}$"

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}\{\underbrace{|k_1,k_2,\dots,k_n\rangle}_{\hat a^+_{k_1}\cdots\hat a^+_{k_n}|0\rangle}\}$$

vacuum: $a_\ell|0\rangle = 0\ \forall\ell$

$$E_{|k_1\dots k_n\rangle} = E_0 + \sum_i\omega_{k_i} \qquad E_0 = \frac12\sum_{k=0}^{N-1}\omega_k$$

$$E_{|k_1\dots k_n\rangle} = \sum_{i=1}^{n}\omega_i$$

↑ n-particle state

time-evolution of fields (Heisenberg picture) $\quad \hat O\longmapsto e^{i\hat Ht}\hat O e^{-i\hat Ht} = \hat O_t$ (time-evol) $\quad$ inf: $\frac{d}{dt}\hat O_t = i[\hat H,\hat O]$

$$\Rightarrow \boxed{\begin{aligned}\hat a^+_t &= e^{i\omega_k t}\hat a^+_k\\ \hat a_t &= e^{-i\omega_k t}\hat a_k\end{aligned}} \Rightarrow \hat{\tilde\varphi}_k(t) = \frac{\hat a_k^+(t) + \hat a_k(t)}{\sqrt2\cdot\sqrt{2\pi R\omega_k}} = \frac{1}{\sqrt{4\pi R\omega_k}}\left(\hat a_k^+\cdot e^{i\omega_k t} + \hat a_k e^{-i\omega_k t}\right)$$

$$\Rightarrow \hat\varphi(x,t) = \cancel{\frac{1}{\sqrt{4\pi R\omega_k}}\sum}\ \boxed{\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{4\pi R\omega_k}}\left(\hat a_k^+\cdot e^{i\omega_k t - \frac{ikx}{R}} + \hat a_k\cdot e^{-i\omega_k t + \frac{ikx}{R}}\right)}$$

→ correlators $\langle 0|\varphi(x_n,t_n)\cdots\varphi(x_1,t_1)|0\rangle$ ⇝ Wick's thm

$t_n > \dots > t_1$