CFT Lecture 13

25.05.11

Free (massive) scalar field on the (Minkowski) cylinder $\mathbb{R}\times S^1$

$\varphi \in C^\infty(\mathbb{R}\times S^1)$, $S = \int dt \int_0^{2\pi R} dx \left(\frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 - \frac{m^2}{2}\varphi^2\right) =$

$= 2\pi R \int dt \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\dot{\tilde\varphi}_k \dot{\tilde\varphi}_{-k} - \frac{\omega_k^2}{2}\tilde\varphi_k\tilde\varphi_{-k}\right)$

Lagrangian picture

Fourier decomposition

$$\varphi(x) = \sum_{k=-\infty}^{\infty} e^{\frac{ikx}{R}}\tilde\varphi_k$$

$$\pi(x) = \sum_{k=-\infty}^{\infty} e^{-\frac{ikx}{R}}\tilde\pi_k \cdot \frac{1}{2\pi R}$$

Frequencies: $\boxed{\omega_k^2 = m^2 + \left(\frac{k}{R}\right)^2}$

Hamiltonian picture

$\Phi = T^* C^\infty(S^1)$ - phase space with coordinates $\varphi(x), \pi(x)$, $x \in \mathbb{R}/_{2\pi R\mathbb{Z}}$, with $\{\pi(x),\varphi(y)\} = \delta_{per}(x-y) = \sum_{n\in\mathbb{Z}} \delta(x-y-2\pi R n)$

or: $\tilde\varphi_k, \tilde\pi_k$, $k\in\mathbb{Z}$, with $\{\tilde\pi_k, \tilde\varphi_l\} = \delta_{kl}$

$$H = \int_0^{2\pi R} dx \left(\frac{1}{2}\pi^2 + (\partial_x\varphi)^2 + \frac{m^2}{2}\varphi^2\right) = \sum_{k=-\infty}^{\infty}\left(\frac{\tilde\pi_k\tilde\pi_{-k}}{2\cdot 2\pi R} + \frac{\omega_k^2}{2}\cdot 2\pi R\,\tilde\varphi_k\tilde\varphi_{-k}\right)$$

$$\Phi = \bigoplus_{k\in\mathbb{Z}} \underbrace{\Phi_{Harm.osc.}}_{T^*\mathbb{R}}, \quad H = \sum_{k\in\mathbb{Z}} H_{Harm.osc.,\,\omega_k}$$

Creation / annihilation operators

$$\boxed{\begin{aligned}\hat a_k &= \frac{1}{\sqrt2}\left(\sqrt{2\pi R\omega_k}\,\hat{\tilde\varphi}_k + \frac{i\hat{\tilde\pi}_{-k}}{\sqrt{2\pi R\,\omega_k}}\right)\\ \hat a_k^+ &= \frac{1}{\sqrt2}\left(\sqrt{2\pi R\omega_k}\,\hat{\tilde\varphi}_{-k} - \frac{i\hat{\tilde\pi}_k}{\sqrt{2\pi R\omega_k}}\right)\end{aligned}}$$

$[\hat a_k, \hat a_l^+] = \delta_{kl}$
$[\hat a_k, \hat a_l] = 0$
$[\hat a_k^+, \hat a_l^+] = 0$

can. comm. rel.
$\left(\Leftrightarrow [\hat\pi_k, \hat\varphi_l] = -i\delta_{kl},\ [\hat\varphi_k, \hat\varphi_l] = 0,\ [\hat\pi_k, \hat\pi_l] = 0\right)$

$$\hat H_{canonical} = \sum_{k=-\infty}^{\infty} \frac{1}{2}\left(\hat a_k^+\hat a_k + \hat a_k\hat a_k^+\right)\cdot\omega_k \qquad [\hat H, \hat a_k^\pm] = \pm\omega_k\hat a_k^\pm$$

Normal ordering

$$\underbrace{X}_{\substack{\text{monomial in}\\ \{\hat a_k\},\{\hat a_k^+\}}} \longmapsto \underbrace{:X:}_{\substack{\text{reshuffled monomial}\\ \text{with } \hat a\text{'s placed to the right}\\ \text{and } \hat a^+\text{'s - to the left}}}$$

e.g. $:\hat a_k \hat a_l^+ \hat a_m^+ \hat a_n: = \hat a_l^+\hat a_m^+\hat a_k\hat a_n$

$:\ldots:$ is extended by $\mathbb{C}$-linearity to $\underbrace{\text{Free Ass Alg}_{\mathbb{C}}(\{\hat a_k\}_{k\in\mathbb{Z}}, \{\hat a_k^+\}_{k\in\mathbb{Z}})}$

Rem: • Normal ordering is not well-defined on $(-\shortparallel-)\Big/\left(\begin{smallmatrix}[\hat a_k, \hat a_l^+] = \delta_{kl}\\ [\hat a, \hat a] = 0\\ [\hat a^+, \hat a^+] = 0\end{smallmatrix}\right) \subseteq End(\mathcal{H})$

• Normal ordering defines a quantization map

$$Fun(\Phi) \longrightarrow End(\mathcal{H})$$

Normally-ordered Hamiltonian:

$$:\hat{H}: = \sum_{k=-\infty}^{\infty} \hat{a}^+_k \hat{a}_k \cdot \omega_k \qquad (\text{differs from } \hat{H}_{can} \text{ by infinite constant})$$

$\hat{H}$-eigenstates: $|k_1 \dots k_n\rangle = \hat{a}^+_{k_1} \dots \hat{a}^+_{k_n} |0\rangle$

$$E_{|k_1 \dots k_n\rangle} = \sum_{i=1}^{n} \omega_{k_i} \quad \leftarrow \text{for } :\hat{H}:$$

Fock space:

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}} \{|k_1 \dots k_n\rangle\} = \bigoplus_{n \geq 0} \underbrace{S^n \mathbb{R}^\infty}$$

"n-particle sector of $\mathcal{H}$"

Rem: for $S^1 \to N$-gon,

$$\mathcal{H} = \bigoplus_{n \geq 0} S^n \mathbb{R}^N$$

Stress-energy tensor $\quad T^\mu{}_\nu = -\partial^\mu \varphi \cdot \partial_\nu \varphi + \delta^\mu_\nu \left(\frac{1}{2} \partial^\lambda \varphi \cdot \partial_\lambda \varphi + \frac{m^2}{2} \varphi^2\right)$

$$T^0{}_0 = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\partial_x \varphi)^2 + \frac{m^2}{2}\varphi^2 \rightsquigarrow H = \int_0^{2\pi R} T^0{}_0 \, dx = \sum_{k=-\infty}^{\infty} \left(\frac{\tilde{\pi}_k \tilde{\pi}_{-k}}{2 \cdot 2\pi R} + \frac{\omega_k^2}{2} \cdot 2\pi R \cdot \tilde{\varphi}_k \tilde{\varphi}_{-k}\right) \Big\} \text{ total energy}$$

$$T^0{}_1 = \dot{\varphi}\, \partial_x \varphi \rightsquigarrow P = \int_0^{2\pi R} T^0{}_1 \, dx = -\sum_{k=-\infty}^{\infty} \frac{ik}{R} \tilde{\pi}_k \tilde{\varphi}_k \Big\} \text{ total momentum}$$

↑ Energy-momentum density $\qquad$ ↑ total Energy-momentum

Quantum version: $\quad :\hat{P}: = \sum_{k=-\infty}^{\infty} \frac{k}{R} \hat{a}^+_k \hat{a}_k$

$$:\hat{H}: |k_1 \dots k_n\rangle = \left(\sum_{i=1}^{n} \omega_{k_i}\right) |k_1 \dots k_n\rangle, \quad :\hat{P}: |k_1 \dots k_n\rangle = \left(\sum_{i=1}^{n} \frac{k_i}{R}\right) |k_1 \dots k_n\rangle$$

$\Downarrow$

Interpretation: state $|k_1 \dots k_n\rangle$ ~ a collection of n identical particles of mass m moving with energies/momenta $\left(\omega_{k_i}, \frac{k_i}{R}\right)$

Rem: $(\omega_{k_i})^2 = m^2 + \left(\frac{k_i}{R}\right)^2$ - relativistic energy/momentum relation

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Time-dependence of fields (Heisenberg picture)

$$\hat{a}^\pm_k(t) = e^{i\hat{H}t} \hat{a}^\pm_k(0) e^{-i\hat{H}t} = e^{\pm i\omega_k t} \hat{a}^\pm_k(0)$$

$$\hat{\varphi}(t,x) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{4\pi R \omega_k}} \left(e^{i(\omega_k t - \frac{k}{R}x)} \hat{a}^+_k(0) + e^{-i(\omega_k t - \frac{k}{R}x)} \hat{a}_k(0)\right)$$

$$\hat{\pi}(t,x) = \sum_{k=-\infty}^{\infty} \frac{i\omega_k}{\sqrt{4\pi R \omega_k}} \left(e^{i(\omega_k t - \frac{k}{R}x)} \hat{a}^+_k(0) - e^{-i(\omega_k t - \frac{k}{R}x)} \hat{a}_k(0)\right)$$

Correlators

Generally, in PI formalism, the correlator is

$$\langle O_1(\underline{x}_1)\cdots O_n(\underline{x}_n)\rangle = \frac{1}{Z}\int D[\Phi(\underline{x})]\, e^{iS[\Phi]}\, O_1(\underline{x}_1)\cdots O_n(\underline{x}_n)$$

in operator formalism on $\Sigma = \mathbb{R}\times M$,

$$\langle O_1(t_1,x_1)\cdots O_n(t_n,x_n)\rangle = \langle 0|\,\underset{\text{time-ordering, putting "later" observables to the right of "earlier" observables}}{\mathrm{T}}\big(\hat{O}_1(t_1,x_1)\cdots\hat{O}_n(t_n,x_n)\big)|0\rangle$$

time-ordering, putting "later" observables to the right of "earlier" observables

Rem: One assumes that correlators are analytic in ε, $0<\mathrm{Re}\,\varepsilon<\pi$ if one makes substitution $t_i \mapsto e^{-i\varepsilon}t_i$, $i=1..n$

In this sense, $\langle 0|\hat{O}_1(t_1,x_1)\cdots\hat{O}_n(t_n,x_n)|0\rangle =$

$$= \langle 0|\hat{O}_1(0,x_1)\,e^{-i\hat{H}(t_1-t_2)}\hat{O}_2(0,x_2)\cdots e^{-i\hat{H}(t_{n-1}-t_n)}\hat{O}_n(0,x_n)|0\rangle$$

converges only for $t_1>t_2>\cdots>t_n$

Correlators for the scalar field on $\mathbb{R}\times S^1$

- $\langle 0|\hat{\varphi}(t,x)|0\rangle = 0$
- $\langle 0|\mathrm{T}\hat{\varphi}(t_1,x_1)\hat{\varphi}(t_2,x_2)|0\rangle = ?$

for $t_1>t_2$, $\hat{\varphi}(t_1,x_1)\hat{\varphi}(t_2,x_2) = :\hat{\varphi}(t_1,x_1)\hat{\varphi}(t_2,x_2): + g(t_1-t_2,x_1-x_2)$

where $g(t,x) = \boxed{\sum_{k=-\infty}^{\infty}\frac{1}{4\pi R\omega_k}e^{-i(\omega_k t-\frac{k}{R}x)}}$ for $t>0$ (converges absolutely for $\mathrm{Im}\,t<0$)

defining $g(-t,x) := g(t,-x)$ for $t>0$,

we have: $\boxed{\mathrm{T}\hat{\varphi}(t_1,x_1)\hat{\varphi}(t_2,x_2) = :\hat{\varphi}(t_1,x_1)\hat{\varphi}(t_2,x_2): + g(t_1-t_2,x_1-x_2)}$ $\forall t_1,t_2$

Thus $\boxed{\langle 0|\mathrm{T}\hat{\varphi}(t_1,x_1)\hat{\varphi}(t_2,x_2)|0\rangle = g(t_1-t_2,x_1-x_2)}$

Rem g satisfies:
- $(\partial_t^2-\partial_x^2+m^2)\,g(t,x)=0$
- $\partial_x g(0,x)=0$, $\partial_t|_{t=+0}(t,x) = -\frac{i}{2}\delta_{per}(x)$
- symmetry $g(t,x) = g(-t,-x)$
- g is (logarithmically) divergent as $t,x\to 0$

- $\langle 0|\mathrm{T}\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4|0\rangle = ?$ Notation: $\hat{\varphi}_i = \hat{\varphi}(t_i,x_i)$, $g_{ij} = g(t_i-t_j,x_i-x_j)$

$$\boxed{\begin{aligned}\mathrm{T}\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4 &= :\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4: + g_{12}:\hat{\varphi}_3\hat{\varphi}_4: + g_{13}:\hat{\varphi}_2\hat{\varphi}_4: + g_{14}:\hat{\varphi}_2\hat{\varphi}_3: + g_{23}:\hat{\varphi}_1\hat{\varphi}_4: + g_{24}:\hat{\varphi}_1\hat{\varphi}_3: +\\ &+ g_{34}:\hat{\varphi}_1\hat{\varphi}_2: + g_{12}g_{34} + g_{13}g_{24} + g_{14}g_{23}\end{aligned}}$$

Thus $\boxed{\langle 0| T\,\hat\varphi_1\hat\varphi_2\hat\varphi_3\hat\varphi_4\,|0\rangle = g_{12}g_{34} + g_{13}g_{24} + g_{14}g_{23}}$

Wick's theorem

$$T\hat\varphi_1\cdots\hat\varphi_n = \sum_{k=0}^{[n/2]} \sum_{\substack{\text{subsets}\\ S\subset\{1\ldots n\}\\ \#S=2k}} \sum_{\substack{\text{ways to split } S \text{ into pairs}\\ S=\bigcup_{m=0}^{k}\{i_m,j_m\}}} :\prod_{p\in\{1\ldots n\}\setminus S}\hat\varphi_p :\; g_{i_1j_1}\cdots g_{i_kj_k}$$

Rem • $\langle 0|T\hat\varphi_1\cdots\hat\varphi_n|0\rangle$ is zero if n is odd

and for $n=2m$, it is a sum of $(2m-1)!!$ terms, each a product of m g's

• These correlators can also be computed in PI formalism, using Wick's thm for momenta of a Gaussian distribution

free massive scalar in $\mathbb{R}^{p,1}$ — momentum $k\in\mathbb{Z} \rightsquigarrow \vec k\in(\mathbb{R}^p)^\vee$

one introduces $\hat a_{\vec k}, \hat a^+_{\vec k}$, $[\hat a_{\vec k}, \hat a^+_{\vec k'}] = \delta(\vec k-\vec k')$ — continuum collection of oscillators with $\omega_{\vec k} = m^2+(\vec k)^2$

n-particle states: $\int d\vec k_1\cdots d\vec k_n \underbrace{\psi(\vec k_1,\ldots,\vec k_n)}_{\substack{n\text{-particle}\\ \text{wave-function}\\ (\text{symmetric in } \vec k_i)}} \underbrace{\hat a^+_{\vec k_1}\cdots\hat a^+_{\vec k_n}|0\rangle}_{\substack{|\vec k_1\cdots\vec k_n\rangle - :\hat H:\text{-eigenstate}\\ \text{with } E=\sum_i \omega_{k_i}}}$

Fock space: $\mathcal{H} = \bigoplus_{n\geq 0} L_2(\underbrace{(\mathbb{R}^p)^\vee\oplus\cdots\oplus(\mathbb{R}^p)^\vee}_{n})^{S_n}$

Wick's thm applies with $g(t,\vec x) \sim \int \frac{d^p\vec k}{\omega_{\vec k}} e^{-i(\omega_{\vec k}t-(\vec k,\vec x))}$

(may be computed explicitly in terms of Bessel functions)

Massless scalar on $\mathbb{R}\times S^1$

$$S = 2\pi R\int dt\; \tfrac12(\dot{\varphi}_0)^2 + \sum_{k=-\infty}^{\infty}\left(\tfrac12 \dot{\tilde\varphi}_k\dot{\tilde\varphi}_{-k} - \frac{\omega_k^2}{2}\tilde\varphi_k\tilde\varphi_{-k}\right),\quad \omega_k = \frac{|k|}{R}$$

$$:\hat H: = \frac{(\hat{\tilde\Pi}_0)^2}{2\cdot 2\pi R} + \sum_{k\neq 0}\hat a_k^+\hat a_k\cdot\frac{|k|}{R}$$

$\begin{pmatrix}\text{massless scalar}\\ \text{on } \mathbb{R}\times S^1\end{pmatrix} = \begin{pmatrix}\text{massless}\\ \text{free particle}\\ \text{on } \mathbb{R}\end{pmatrix} + \begin{pmatrix}\text{harmonic}\\ \text{oscillators}\\ \text{with } \omega_k=\frac{|k|}{R}\end{pmatrix}_{k\neq 0}$

$\hat H$-eigenstates: $|k_1\cdots k_n; \tilde\Pi_0\rangle = \hat a^+_{k_1}\cdots\hat a^+_{k_n}|\tilde\Pi_0\rangle$, $E = \boxed{\frac{\tilde\Pi_0^2}{2\cdot 2\pi R} + \sum_{i=1}^n\frac{|k_i|}{R}}$

pseudo-vacuum

"center of mass momentum"

Hilbert space:

$$\mathcal{H} = \underbrace{\mathcal{H}_{free}}_{L_2(\mathbb{R})}\otimes\left(\bigoplus_{n\geq 0} S^n\mathbb{R}^\infty\right)$$

Left & right-movers

$$\hat\varphi(t,x) = \hat{\tilde\varphi}_0 + \frac{\hat{\tilde\Pi}_0}{2\pi R}t + \sum_{k\neq 0}\frac{1}{\sqrt{4\pi|k|}}\left(\hat a_k^+ e^{i\frac{|k|t-kx}{R}} + \hat a_k e^{-i\frac{|k|t-kx}{R}}\right) =$$

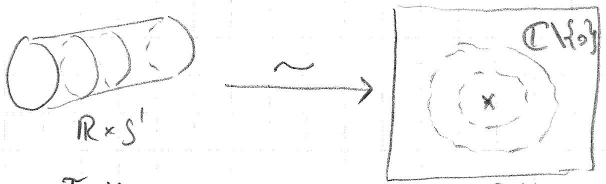
$$= \hat{\tilde\varphi}_0 + \frac{\hat{\tilde\Pi}_0}{2\pi R}t + \sum_{k\neq 0}\frac{1}{\sqrt{4\pi|k|}}\left(\hat b_k e^{i\frac{k}{R}(t+x)} + \hat c_k e^{i\frac{k}{R}(t-x)}\right)$$

where for $k>0$, $\hat b_k = \hat a_{-k}^+$, $\hat b_{-k} = \hat a_{-k}$ - creation/annihilation of "right-movers"
$\hat c_k = \hat a_k^+$, $\hat c_{-k} = \hat a_k$ - —— " —— "left-movers"

$$\begin{cases} [b_k, b_{k'}] = -\delta_{k,-k'}\,\mathrm{sign}(k) \\ [c_k, c_{k'}] = -\delta_{k,-k'}\,\mathrm{sign}(k) \\ [b_k, c_{k'}] = 0 \end{cases} \qquad \begin{matrix} (b_k)^+ = b_{-k} \\ (c_k)^+ = c_{-k} \end{matrix}$$

Euclidean version

- by Wick rotation: $t_{Mink} = -i\tau_{Eucl}$, s.t. $e^{-i\hat H t} \mapsto e^{-\hat H\tau}$



$\mathbb{R}\times S^1$

τ, x

$z = e^{\frac{\tau+ix}{R}}$

$\bar z = e^{\frac{\tau-ix}{R}}$

$$\hat\varphi(\tau,x) = \hat{\tilde\varphi}_0 - i\tau\frac{\hat{\tilde\Pi}_0}{2\pi R} + \sum_{k\neq 0}\frac{1}{\sqrt{4\pi|k|}}\left(\hat b_k e^{\frac{k}{R}(\tau+ix)} + \hat c_k e^{\frac{k}{R}(\tau-ix)}\right)$$

$$= \boxed{\hat{\tilde\varphi}_0 - \frac{i\hat{\tilde\Pi}_0}{2\pi}\log|z| + \sum_{k\neq 0}\frac{1}{\sqrt{4\pi|k|}}\left(\hat b_k z^k + \hat c_k \bar z^k\right)}$$

Eucl. propagator

$$R\,\varphi(z_1,\bar z_1)\,\varphi(z_2,\bar z_2) = :\varphi(z_1,\bar z_1)\,\varphi(z_2,\bar z_2): + g(z_1,\bar z_1; z_2,\bar z_2)$$

radial ordering

with $\boxed{g = -\frac{1}{2\pi}\log|z_1-z_2| + C}$ $(*)$

infinite constant due to $\langle 0|\hat\varphi_0\hat\varphi_0|0\rangle$ term

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Rem: from PI formalism, one gets $(*)$ as the Green's function for $\Delta = 4\partial\bar\partial$, with C an arbitrary constant

Rem: Normal ordering we use also puts $\hat{\tilde\Pi}_0$ to the right of $\hat{\tilde\varphi}_0$

Stress-energy tensor

$T(z) = \partial\varphi\cdot\partial\varphi \rightsquigarrow \hat T(z) = :\partial\hat\varphi(z)\cdot\partial\hat\varphi(z):$

$\bar T(\bar z) = \bar\partial\varphi\cdot\bar\partial\varphi \rightsquigarrow \hat{\bar T}(\bar z) = :\bar\partial\hat\varphi(z)\cdot\bar\partial\hat\varphi(z):$

Examples of OPEs for massless scalar (via Wick's thm)

$$R\left(\hat{\varphi}(z_1,\bar{z}_1)\,\hat{\varphi}(z_2,\bar{z}_2)\right) = -\frac{1}{2\pi}\log|z_1 - z_2| + \text{regular terms}$$

$$R\left(\partial\hat{\varphi}(z_1)\,\partial\hat{\varphi}(z_2)\right) = -\frac{1}{4\pi}\frac{1}{(z_1-z_2)^2} + \text{reg.} \quad ; \quad R\left(\partial\hat{\varphi}(z_1)\,\bar{\partial}\hat{\varphi}(\bar{z}_2)\right) = \text{reg.}$$

$$R\left(\hat{T}(w)\,\hat{\varphi}(z)\right) = -\frac{1}{2\pi}\frac{1}{w-z}\partial\hat{\varphi}(z) + \text{reg.}$$

$$R\left(\hat{T}(w)\,\partial\hat{\varphi}(z)\right) = -\frac{1}{2\pi}\left(\frac{1}{(w-z)^2}\partial\hat{\varphi}(z) + \frac{1}{w-z}\partial^2\hat{\varphi}(z)\right) + \text{reg.}$$

$$R\left(\hat{T}(w)\,\hat{T}(z)\right) = \frac{1}{8\pi^2}\frac{1}{(w-z)^4} - \frac{2}{2\pi}\frac{1}{(w-z)^2}\hat{T}(z) - \frac{1}{2\pi}\frac{1}{w-z}\partial\hat{T}(z) + \text{reg.}$$