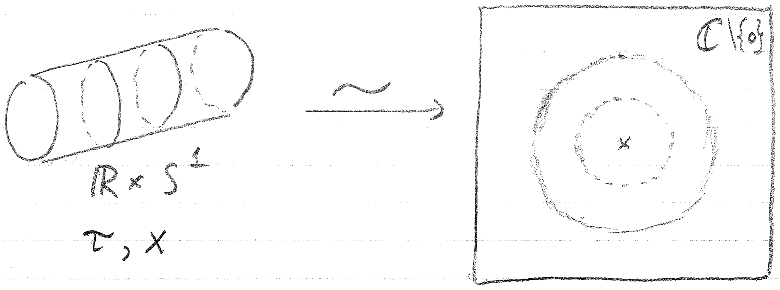
CFT Lecture 14

01.06.11

2D Massless scalar, Euclidean version

Wick rotation (Minkowski $\to$ Euclidean): $t_{Mink} = -i\tau_{Eucl}$, s.t. $e^{-i\hat{H}t} \mapsto e^{-\hat{H}\tau}$



$z = e^{\frac{\tau+ix}{R}}$

$\bar{z} = e^{\frac{\tau-ix}{R}}$

Field:

$$\hat{\varphi}(\tau,x) = \hat{\varphi}_0 - \frac{i\hat{\tilde{\Pi}}_0}{2\pi R}\tau + \sum_{k\neq 0}\frac{\hat{b}_k e^{\frac{k}{R}(\tau+ix)} + \hat{c}_k e^{\frac{k}{R}(\tau-ix)}}{\sqrt{4\pi|k|}} =$$

$$\boxed{= \hat{\tilde{\varphi}}_0 - \frac{i\hat{\tilde{\Pi}}_0}{2\pi}\log|z| + \sum_{k\neq 0}\frac{1}{\sqrt{4\pi|k|}}(\hat{b}_k z^k + \hat{c}_k \bar{z}^k)}$$

$$\left(= \hat{\tilde{\varphi}}_0 + \hat{\varphi}^{hol}(z) + \hat{\varphi}^{antihol}(\bar{z}) \right)$$

Euclidean propagator

$$\mathcal{R}\,\hat{\varphi}(z_1,\bar{z}_1)\,\hat{\varphi}(z_2,\bar{z}_2) = :\hat{\varphi}(z_1,\bar{z}_1)\,\hat{\varphi}(z_2,\bar{z}_2): + g(z_1,\bar{z}_1; z_2,\bar{z}_2)$$

radial ordering

with $\boxed{g = -\frac{1}{2\pi}\log|z_1-z_2| + C}$ $(*)$

infinite constant due to $\langle 0|\hat{\tilde{\varphi}}_0\hat{\tilde{\varphi}}_0|0\rangle$ term

Rem: Here the normal ordering
- puts $\hat{a}_k^+$ to the left of $\hat{a}_k$
- puts $\hat{\tilde{\varphi}}_0$ to the left of $\hat{\tilde{\Pi}}_0$
- eliminates the $(\hat{\tilde{\varphi}}_0)^n$ term, so that $\langle 0| :\ldots: |0\rangle = 0$

Rem: In PI formalism, one gets $(*)$ as the Green's function for $\Delta = 4\partial\bar{\partial}$ with C an arbitrary constant.

Stress-energy tensor

$T(z) = \partial\varphi(z)\cdot\partial\varphi(z)$

$\bar{T}(\bar{z}) = \bar{\partial}\varphi(\bar{z})\cdot\bar{\partial}\varphi(\bar{z})$

$\rightsquigarrow$

$\hat{T}(z) = :\partial\hat{\varphi}(z)\cdot\partial\hat{\varphi}(z):$

$\hat{\bar{T}}(\bar{z}) = :\bar{\partial}\hat{\varphi}(\bar{z})\,\bar{\partial}\hat{\varphi}(\bar{z}):$

(normal ordering ensures that $\langle T\rangle = \langle\bar{T}\rangle = 0$ removing the infinite constant)

Operator product expansions (OPE) generally

One writes $\mathcal{O}_1(x)\cdot\mathcal{O}_2(y) \sim \sum C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_k}(x,y)\,\mathcal{O}_k(y)$ if such substitution does not change correlators $\langle\ldots\mathcal{O}_1(x)\mathcal{O}_2(y)\ldots\rangle$

local fields $\mathcal{O}_k$; some functions

The singular part of OPE:

$$\mathcal{O}_1(x)\cdot\mathcal{O}_2(y) \underset{x\to y}{\sim} \sum_k C^{sing}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_k}(x,y)\,\mathcal{O}_k(y) + \text{regular terms}$$

OPE

In operator formalism on $\mathbb{C}\setminus\{0\}$ ("radial quantization")

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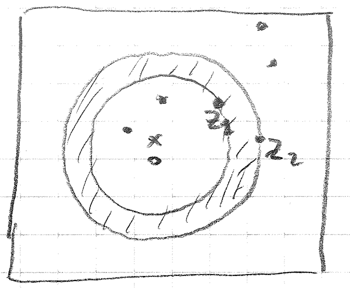
$$\mathcal{R}\, \hat{O}_1(z_1,\bar z_1)\, \hat{O}_2(z_2,\bar z_2) = \sum_k C_{\hat O_1,\hat O_2,\hat O_k}(z_1,\bar z_1; z_2,\bar z_2)\, \hat O_k(z_2,\bar z_2)$$

in the sense that such a substitution can be made in correlators

$$\langle 0|\mathcal{R}(\cdots\, \hat O_1(z_1,\bar z_1)\, \hat O_2(z_2,\bar z_2)\cdots)|0\rangle$$

if all other fields are outside the annulus $\{\min(|z_1|,|z_2|)\le r\le \max(|z_1|,|z_2|)\}$



Case of scalar field

- $$\mathcal{R}\,\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w) = \left(-\frac{1}{2\pi}\log|z-w| + C\right)\hat 1 + \underbrace{:\hat\varphi(z,\bar z)\hat\varphi(w,\bar w):}$$

$$:\varphi(w,\bar w)\varphi(w,\bar w): + \sum_{n>0}\frac{1}{n!}(z-w)^n :\partial^n\hat\varphi(w)\cdot\hat\varphi(w,\bar w): + \sum_{n>0}\frac{1}{n!}(\bar z-\bar w)^n :\bar\partial^n\hat\varphi(\bar w)\cdot\hat\varphi(w,\bar w):$$

i.e. $$\mathcal{R}\,\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w) \underset{z\to w}{\sim} \left(-\frac{1}{2\pi}\log|z-w|+C\right)\hat 1 + \text{reg.}$$

- $$\mathcal{R}\,\partial\hat\varphi(z)\,\partial\hat\varphi(w) = -\frac{1}{4\pi}\frac{1}{(z-w)^2}\hat 1 + \text{reg.}$$

likewise: $$\mathcal{R}\,\bar\partial\hat\varphi(\bar z)\,\bar\partial\hat\varphi(\bar w) = -\frac{1}{4\pi}\frac{1}{(\bar z-\bar w)^2}\hat 1 + \text{reg.}$$

$$\mathcal{R}\,\partial\hat\varphi(z)\cdot\bar\partial\hat\varphi(\bar w) = \text{reg.}$$

- $$\mathcal{R}\,\underbrace{\hat T(z)}_{:\partial\hat\varphi(z)\cdot\partial\hat\varphi(z):}\,\hat\varphi(w,\bar w) = -\frac{1}{2\pi}\frac{1}{z-w}\partial\hat\varphi(w) + \text{reg.}$$

$\leftarrow$ consequence of Wick's thm

$$\mathcal{R}\,\hat T(z)\cdot\partial\hat\varphi(w) = -\frac{1}{2\pi}\left(\frac{1}{(z-w)^2}\partial\hat\varphi(w) + \frac{1}{z-w}\partial^2\hat\varphi(w)\right) + \text{reg.}$$

- $$\mathcal{R}\,\hat T(z)\cdot\hat T(w) = \frac{1}{8\pi^2}\frac{1}{(z-w)^4} - \frac{2}{2\pi}\frac{1}{(z-w)^2}\hat T(w) - \frac{1}{2\pi}\frac{1}{z-w}\partial\hat T(z) + \text{reg.}$$

< General story >

Primary fields · with transformation rule $\quad z\mapsto w(z)$

$$\Phi(z,\bar z)\mapsto \Phi'(w,\bar w) = \Phi(z,\bar z)\left(\frac{\partial z}{\partial w}\right)^h\left(\frac{\partial\bar z}{\partial\bar w}\right)^{\bar h}$$

(i.e. Φ transforms like a section of the complex line bundle $\left((T^*_{hol})^{\otimes h}\otimes(T^*_{anti\text{-}hol})^{\otimes\bar h}\right)\Sigma$)

Φ is said to be a primary field of conformal weight $(h,\bar h)\in\mathbb{R}^2$

Infinitesimally: $\quad z\mapsto z+\varepsilon$

$$\Phi(z,\bar z)\mapsto \Phi - \varepsilon\partial\Phi - \bar\varepsilon\bar\partial\Phi - h\Phi\cdot\partial\varepsilon - \bar h\,\Phi\cdot\bar\partial\bar\varepsilon$$

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Correlators of primary fields in a CFT

$$\langle \Phi_1(z_1,\bar z_1)\cdots\Phi_n(z_n,\bar z_n)\rangle = \prod_{k=1}^{n}\left(\left(\frac{\partial w}{\partial z}\right)^{-h_k}\left(\frac{\partial \bar w}{\partial \bar z}\right)^{-\bar h_k}\right)\Bigg|_{z_k,\bar z_k}\cdot\langle \Phi_1(w_1,\bar w_1)\cdots\Phi_n(w_n,\bar w_n)\rangle$$

for any holom. map $\mathbb{CP}^1 \to \mathbb{CP}^1$, $z \mapsto w$ (from (naive) PI formalism)

This implies:

- $$\langle \Phi(z,\bar z)\rangle = \begin{cases} \text{const}, & \text{if } h=\bar h=0 \\ 0 & \text{otherwise}\end{cases}$$
- $$\langle \Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\rangle = \begin{cases} \dfrac{C_{12}}{z_{12}^{2h}\,\bar z_{12}^{2\bar h}}, & \text{if } h_1=h_2,\ \bar h_1=\bar h_2 \\ 0 & \text{otherwise}\end{cases} \qquad z_{12}=z_1-z_2$$
- $$\langle \Phi_1(z_1,\bar z_1)\Phi_2(z_2,\bar z_2)\Phi_3(z_3,\bar z_3)\rangle = C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}\,z_{23}^{h_2+h_3-h_1}\,z_{13}^{h_1+h_3-h_2}}\cdot\frac{1}{\bar z_{12}^{\bar h_1+\bar h_2-\bar h_3}\,\bar z_{23}^{\bar h_2+\bar h_3-\bar h_1}\,\bar z_{13}^{\bar h_3+\bar h_1-\bar h_2}}$$
- $$\langle \Phi_1\Phi_2\Phi_3\Phi_4\rangle = f(x,\bar x)\prod_{i<j} z_{ij}^{-(h_i+h_j)+\frac{h}{3}}\cdot \bar z_{ij}^{-(\bar h_i+\bar h_j)+\frac{\bar h}{3}}$$
  $$h=\sum_{i=1}^4 h_i,\quad \bar h=\sum_{i=1}^4 \bar h_i,\quad x=\frac{z_{12}z_{34}}{z_{13}z_{24}} \text{ - cross-ratio}$$
  f cannot be determined from global conformal symmetry

Action of a conformal vector field on quantum fields in operator formalism (radial quantization)

Reminder: in Ham. mechanics, a symmetry with Nöther charge I acts on phase-space by $\{I,\cdot\}$

In quantum theory: $\delta_{sym}\hat\Phi(z,\bar z) = [\hat I_{sym},\hat\Phi(z,\bar z)]$

Nöther charge for a conformal v.f. $\varepsilon\partial+\bar\varepsilon\bar\partial$: $\hat J_\varepsilon = -i\left(\oint \hat T\varepsilon\, dz - \oint \hat{\bar T}\bar\varepsilon\, d\bar z\right)$

Thus $\delta_\varepsilon\hat\Phi(z,\bar z) = [\hat J_\varepsilon,\hat\Phi(z,\bar z)] =$

$$= -i\oint \mathcal{R}\big(\hat T(w)\hat\Phi(z,\bar z)\big)\varepsilon(w)\,dw - \mathcal{R}\big(\hat{\bar T}(\bar w)\hat\Phi(z,\bar z)\big)\bar\varepsilon(\bar w)\,d\bar w$$

by Cauchy thm can deform the contour

$\Rightarrow$ $\delta_\varepsilon\hat\Phi$ can be found from the singular parts of $\hat T\hat\Phi$ and $\hat{\bar T}\hat\Phi$ OPEs

$$\left.\begin{aligned}\mathcal{R}\,\hat T(w)\hat\Phi(z,\bar z) &= -\frac{1}{2\pi}\frac{h}{(w-z)^2}\hat\Phi(z,\bar z) - \frac{1}{2\pi}\frac{1}{w-z}\partial\hat\Phi(z,\bar z) + \text{reg.}\\ \mathcal{R}\,\hat{\bar T}(w)\hat\Phi(z,\bar z) &= -\frac{1}{2\pi}\frac{\bar h}{(\bar w-\bar z)^2}\hat\Phi(z,\bar z) - \frac{1}{2\pi}\frac{1}{\bar w-\bar z}\bar\partial\hat\Phi(z,\bar z) + \text{reg.}\end{aligned}\right\}\Rightarrow$$

$\Rightarrow$ $\delta_\varepsilon\hat\Phi(z,\bar z) = \left(-\varepsilon\partial\hat\Phi - h\,\partial\varepsilon\cdot\hat\Phi\right) + \left(-\bar\varepsilon\bar\partial\hat\Phi - \bar h\,\bar\partial\bar\varepsilon\cdot\hat\Phi\right)$ - inf. transformation rule for a <u>primary</u> field

T-T OPE & commutators of Nöther charges $\hat{J}_\varepsilon$

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One postulates the ansatz

$$\begin{cases} R\,\hat{T}(w)\hat{T}(z) = \dfrac{c}{8\pi^2}\dfrac{1}{(w-z)^4} - \dfrac{2}{2\pi}\dfrac{\hat{T}(z)}{(w-z)^2} - \dfrac{1}{2\pi}\dfrac{\partial\hat{T}(z)}{w-z} + \text{reg.} \\ R\,\hat{\bar{T}}(\bar{w})\hat{\bar{T}}(\bar{z}) = \dfrac{\bar{c}}{8\pi^2}\dfrac{1}{(\bar{w}-\bar{z})^4} - \dfrac{2}{2\pi}\dfrac{\hat{\bar{T}}(\bar{z})}{(\bar{w}-\bar{z})^2} - \dfrac{1}{2\pi}\dfrac{\bar{\partial}\hat{\bar{T}}(\bar{z})}{\bar{w}-\bar{z}} + \text{reg.} \\ R\,\hat{T}(w)\hat{\bar{T}}(z) = \text{reg.} \end{cases}$$

(in fact, the ansatz can be derived from 1) $w \leftrightarrow z$ symmetry 2) requirement $[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\hat{\Phi}(z,\bar{z}) = \delta_{[\varepsilon_1,\varepsilon_2]}\hat{\Phi}(z,\bar{z})$

3) scaling dimension 2 for T implies
$\langle T(0)T(z)\rangle = \dfrac{\text{const}}{z^4}$)

$(c, \bar{c})$ - central charges

$\hat{T}, \hat{\bar{T}}$ are not primary fields!

$$\delta_\varepsilon \hat{T}(z) = -i\oint dw\, R\,\hat{T}(w)\hat{T}(z)\cdot\varepsilon(w) = -\varepsilon\cdot\partial\hat{T} - 2(\partial\varepsilon)\cdot\hat{T} + \frac{c}{24\pi}\partial^3\varepsilon \qquad (*)$$

i.e. $\hat{T}$ transforms like a $(2,0)$ primary field under Möbius transformations, but gets a correction for more general conf. vector fields

Finite version of $(*)$: $\quad z \longmapsto w$

$$T(z) \longmapsto T'(w) = \left(\frac{\partial w}{\partial z}\right)^{-2} T(z) + \frac{c}{24\pi}\, S(z,w)$$

$$S(z,w) = \frac{\partial_w z\cdot\partial_w^3 z - \frac{3}{2}(\partial_w^2 z)^2}{(\partial_w z)^2} \quad \text{- "Schwartzian derivative"}$$

Commutators of $\hat{J}_\varepsilon$ define $\hat{I}_\varepsilon = -i\oint dz\,\hat{T}(z)\varepsilon(z)$, $\hat{\bar{I}}_{\bar{\varepsilon}} = i\oint d\bar{z}\,\hat{\bar{T}}(\bar{z})\bar{\varepsilon}(\bar{z})$

s.t. $\hat{J}_\varepsilon = \hat{I}_\varepsilon + \hat{\bar{I}}_{\bar{\varepsilon}}$

$$[\hat{I}_{\varepsilon_1}, \hat{I}_{\varepsilon_2}] =$$

$$= -\underbrace{\left(\oint dz \oint dw - \oint dz \oint dw\right)}_{\text{deform contours}} R\,T(w)T(z)\cdot\varepsilon_1(z)\varepsilon_2(w) \;\text{\textcircled{=}}$$

$$\oint dz \oint dw$$

$$\text{\textcircled{=}}\; \hat{I}_{\varepsilon_1\partial\varepsilon_2 - \varepsilon_2\partial\varepsilon_1} + \frac{ic}{24\pi}\oint dz\,\varepsilon_1(z)\,\partial^3\varepsilon_2(z)$$

Likewise $\quad [\hat{\bar{I}}_{\bar{\varepsilon}_1}, \hat{\bar{I}}_{\bar{\varepsilon}_2}] = \hat{\bar{I}}_{\bar{\varepsilon}_1\bar{\partial}\bar{\varepsilon}_2 - \bar{\varepsilon}_2\bar{\partial}\bar{\varepsilon}_1} - \dfrac{i\bar{c}}{24\pi}\oint d\bar{z}\,\bar{\varepsilon}_1(\bar{z})\,\bar{\partial}^3\bar{\varepsilon}_2(\bar{z})$

$$[\hat{I}_{\varepsilon_1}, \hat{\bar{I}}_{\bar{\varepsilon}_2}] = 0$$

Thus $\quad \text{conf}(\mathbb{C}\setminus\{0\}) \longrightarrow \text{End}(\mathcal{H})$ is a projective representation

$$\varepsilon\partial + \bar{\varepsilon}\bar{\partial} \longmapsto \hat{J}_\varepsilon = \hat{I}_\varepsilon + \hat{\bar{I}}_{\bar{\varepsilon}}$$

Virasoro algebra

$$l_n = -z^{n+1}\partial \rightsquigarrow \hat{I}_{-z^{n+1}} = i\oint dz\, z^{n+1}\,\hat{T}(z) =: \hat{L}_n$$
$$\bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \rightsquigarrow \hat{\bar{I}}_{-\bar{z}^{n+1}} = -i\oint d\bar{z}\, \bar{z}^{n+1}\,\hat{\bar{T}}(\bar{z}) =: \hat{\bar{L}}_n$$

Commutation relations:

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}\delta_{n,-m}(n^3-n)\cdot\hat{1}$$
$$[\hat{\bar{L}}_n, \hat{\bar{L}}_m] = (n-m)\hat{\bar{L}}_{n+m} + \frac{\bar{c}}{12}\delta_{n,-m}(n^3-n)\cdot\hat{1}$$

$\leftarrow \mathrm{Span}(\{\hat{L}_n\}, \hat{1}) = Vir$ - Virasoro algebra = unique central extension of Witt algebra

i.e. the space of states $\mathcal{H}$ of a CFT is a representation of $Vir \oplus \overline{Vir}$

- $$\hat{T}(z) = -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n, \quad \hat{\bar{T}}(\bar{z}) = -\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\bar{z}^{-n-2}\hat{\bar{L}}_n$$

- in terms of cylinder, $z = e^{\frac{\tau+ix}{R}}$,
$$\begin{cases}\hat{H} = \frac{1}{R}(\hat{L}_0 + \hat{\bar{L}}_0) & \text{- Hamiltonian (energy)}\\ \hat{P} = \frac{i}{R}(\hat{L}_0 - \hat{\bar{L}}_0) & \text{- total momentum}\end{cases}$$
$\{\hat{L}_n, \hat{\bar{L}}_n\}$ are integrals of motion assoc. to conformal symmetry

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Conformal Ward identities

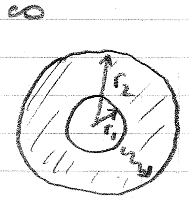
$$\langle T(z)\underbrace{\Phi_1(z_1,\bar{z}_1)\cdots\Phi_n(z_n,\bar{z}_n)}_{\text{primary fields}}\rangle = -\frac{1}{2\pi}\sum_{k=1}^{n}\left(\frac{h_k}{(z-z_k)^2} + \frac{1}{z-z_k}\frac{\partial}{\partial z_k}\right)\langle\Phi_1(z_1,\bar{z}_1)\cdots\Phi_n(z_n,\bar{z}_n)\rangle$$

(since $\langle T(z)\Phi_1\cdots\Phi_n\rangle$ is a meromorphic function in z with poles at $z_1,\ldots,z_n$ and with principal parts of Laurent expansions known due to $T\Phi$ OPE)

likewise for $\langle\bar{T}(\bar{z})\Phi_1(z_1,\bar{z}_1)\cdots\Phi_n(z_n,\bar{z}_n)\rangle$

"Segal's picture" (vs. radial quantization formalism)

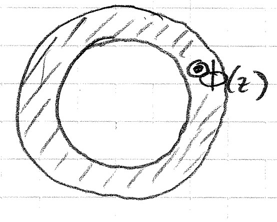
- (quantum) fields at point $z \in \mathbb{CP}^1$ = elements of $\mathcal{H}_z$
- in-states = fields at $z=0$
  out-states = fields at $z=\infty$
- evol. operator for annulus



$$\mathcal{Z} = e^{-\underbrace{\hat{H}}_{\frac{1}{R}(\hat{L}_0+\hat{\bar{L}}_0)}\underbrace{\tau}_{R\log\frac{r_2}{r_1}}} = \boxed{\left(\frac{r_1}{r_2}\right)^{\hat{L}_0+\hat{\bar{L}}_0}}$$

$\hat{H}$ provides a connection allowing to identify $\mathcal{H}_{|z|=r}$ for different r (Heisenberg picture)

- operator associated to a local field = partition function for punctured annulus with puncture decorated by Φ



local Virasoro algebras

$$\mathcal{L}_n^{(z_0)} := i \oint_{z_0} dz\, (z-z_0)^{n+1}\, T(z) \;,\; \bar{\mathcal{L}}_n^{(z_0)} := -i \oint_{z_0} d\bar{z}\, (\bar{z}-\bar{z}_0)^{n+1}\, \bar{T}(\bar{z})$$

So we have $Vir^{(z_k)} \oplus \overline{Vir}^{(z_k)}$ at every puncture z_k, acting on $\text{Fields}^{(z_k)}$

the action is given by terms of $T\Phi$, $\bar{T}\Phi$ OPE:

$$T(z)\,\Phi(z_0,\bar{z}_0) = -\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} (z-z_0)^{-n-2}\, \mathcal{L}_n^{(z_0)} \Phi(z_0,\bar{z}_0)$$

$$\bar{T}(\bar{z})\,\Phi(z_0,\bar{z}_0) = -\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} (\bar{z}-\bar{z}_0)^{-n-2}\, \bar{\mathcal{L}}_n^{(z_0)} \Phi(z_0,\bar{z}_0)$$

- for primary fields, $$\mathcal{L}_n^{(z)} \Phi(z) = \begin{cases} 0 \;, & n \geq 1 \\ h\,\Phi(z) \;, & n = 0 \\ \partial\Phi(z) \;, & n = -1 \\ \text{(regular terms of } T\Phi \text{ OPE)} \;, & n < -1 \end{cases}$$
- for the identity field $\mathbb{1}$, $$\mathcal{L}_{-1}^{(z)} \mathbb{1}^{(z)} = 0 \;,\; \boxed{\mathcal{L}_{-2}^{(z)} \mathbb{1}^{(z)} = (-2\pi)\, T(z)}$$

Fields at a given point z fall into "conformal families"

$\Phi_{(1)}^{(h,\bar{h})}$ primary of weight ($(\mathcal{L}_0,\bar{\mathcal{L}}_0)$ eigenvalue) $(h,\bar{h})$

$$\left.\begin{array}{l} \mathcal{L}_{-1}\Phi_{(1)}^{(h+1,\bar{h})} \;,\; \bar{\mathcal{L}}_{-1}\Phi_{(1)}^{(h,\bar{h}+1)} \\ \mathcal{L}_{-2}\Phi_{(1)},\ \mathcal{L}_{-1}\mathcal{L}_{-1}\Phi_{(1)},\ \bar{\mathcal{L}}_{-2}\Phi_{(1)},\ \bar{\mathcal{L}}_{-1}\bar{\mathcal{L}}_{-1}\Phi_{(1)},\ \mathcal{L}_{-1}\bar{\mathcal{L}}_{-1}\Phi_{(1)} \\ \mathcal{L}_{-3}\Phi_{(1)},\ \mathcal{L}_{-2}\mathcal{L}_{-1}\Phi_{(1)} \;\ldots \\ \ldots \end{array}\right\} \text{descendants}$$

↑

Verma module for $Vir^{(z)} \oplus \overline{Vir}^{(z)}$ generated by the highest weight vector $\Phi_{(1)}$

- there is a special conf. family generated by $\mathbb{1}$
- OPE is an additional structure on $\{\text{Fields}^{(z)}\}_{z\in\mathbb{CP}^1}$ (fusion algebra)
- $\mathcal{L}_{-1}, \bar{\mathcal{L}}_{-1}$ provide a connection on $\begin{array}{c}\text{Fields}\\ \downarrow \\ \mathbb{CP}^1\end{array}$

Conformal Ward identities (revisited)

for a merom. v.f. $\varepsilon\partial$, we have

$$\langle (\mathcal{L}_\varepsilon^{(z_1)} \Phi_1(z_1,\bar{z}_1))\, \Phi_2(z_2,\bar{z}_2) \cdots \Phi_n(z_n,\bar{z}_n)\rangle + \ldots + \langle \Phi_1(z_1,\bar{z}_1) \cdots \Phi_{n-1}(z_{n-1},\bar{z}_{n-1})\, (\mathcal{L}_\varepsilon^{(z_n)} \Phi_n(z_n,\bar{z}_n))\rangle = 0$$

where $\mathcal{L}_\varepsilon^{(z_k)} \Phi_k := -i \oint_{z_k} dz\, \varepsilon(z)\, T(z)\, \Phi_k(z_k,\bar{z}_k)$

for $\{\Phi_k\}$ primary, $\Phi_0 = \mathbb{1}^{(z_0)}$, $\varepsilon\partial = \frac{1}{z-z_0}\frac{\partial}{\partial z}$, we get the standard conf. Ward identity for $\langle T\Phi_1 \cdots \Phi_n\rangle$