Introduction to Conformal Field Theory

Lecture 2

Plan for today: Conformal symmetry $\longrightarrow$ definitions, basic properties, examples of conformal maps

↳ Conformal vector fields in $\mathbb{R}^{p,q}$, $p+q>2$

↳ Special cases: $\mathbb{R}^2$, $\mathbb{R}^{1,1}$, $\mathbb{R}^1$

Conformal symmetry

Let $(\mathcal{M}, g)$ be a (pseudo-) Riemannian mfd.

<u>def</u> a Weyl transformation is:

$$\begin{aligned}(\mathcal{M},g) &\longrightarrow (\mathcal{M},g') \\ x &\longmapsto x \\ g(x) &\longmapsto \Omega(x)\cdot g(x) = g'(x)\end{aligned}$$

where $\Omega(x)$ is an everywhere positive function

<u>def</u> two (pseudo-) Riemannian mfds $(\mathcal{M},g)$, $(\mathcal{M}',g')$ are said to be <u>conformally equivalent</u> if there exists a diffeomorphism $\varphi: M \longrightarrow M'$ s.t. $(\varphi^* g')(x) = \Omega(x)\cdot g(x)$

Then φ is called a <u>conformal map</u> and Ω is the associated <u>conformal factor</u>

<u>Obvious properties</u>:

- composition of conf. maps $(\mathcal{M},g) \xrightarrow{\varphi_1} (\mathcal{M}',g') \xrightarrow{\varphi_2} (\mathcal{M}'',g'')$ is a conf. map; the conf. factors multiply: $\Omega_{\varphi_2\circ\varphi_1} = \varphi_1^*(\Omega_{\varphi_2})\cdot\Omega_{\varphi_1}$

  $\Omega_{\varphi_2\circ\varphi_1}(x) = \Omega_{\varphi_2}(\varphi_1(x))\cdot\Omega_{\varphi_1}(x)$

- Inverse of a conf. map $(M,g) \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} (M',g')$ is also a conf. map with

  $\Omega_{\varphi^{-1}}(x') = \left(\Omega_\varphi(\varphi^{-1}(x))\right)^{-1}$

- Identity map $(\mathcal{M},g) \xrightarrow{id} (\mathcal{M},g)$ is conformal with $\Omega \equiv 1$

<u>def</u> Conformal automorphisms $\varphi: (\mathcal{M},g) \longrightarrow (\mathcal{M},g)$ comprise the <u>conformal group</u> $\mathrm{Conf}(\mathcal{M},g)$

<u>def</u> a <u>conformal structure</u> on $\mathcal{M}$ is a choice of metric modulo Weyl transformations

<informally: a conf. str. is a way to measure angles between tangent vectors>

- for $g\sim g'$ two Weyl-equivalent metrics on $\mathcal{M}$, $\mathrm{Conf}(\mathcal{M},g) \cong \mathrm{Conf}(M,g')$

  <conf. maps are the same, but conf. factors may differ>

i.e. the group $\mathrm{Conf}(\mathcal{M}, g/_\sim)$ depends in fact only on the choice of conf. structure on $\mathcal{M}$

- $\{\text{isometries}\} \hookrightarrow \{\text{conformal maps}\}$, singled out by the property $\Omega \equiv 1$

Examples of conformal maps

- translations and $O(n)$-rotations in Euclidean $\mathbb{R}^n$

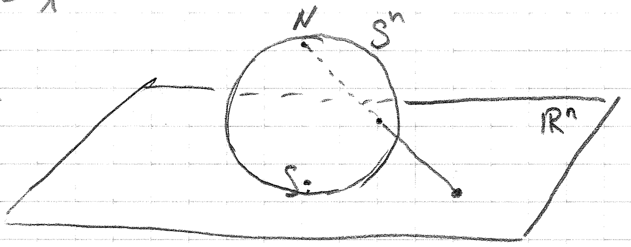
  (or translations and $O(p,q)$-rotations in $\mathbb{R}^{p,q}$ with $g=(dx^1)^2+\dots+(dx^p)^2-(dx^{p+1})^2-\dots-(dx^{p+q})^2$)

  i.e. $\underbrace{O(n)\ltimes\mathbb{R}^n}_{\text{Poincaré group}} \subset \mathrm{Conf}(\mathbb{R}^n)$ - they are exactly the isometries, $\Omega\equiv 1$

- dilatations $\begin{array}{ccc}\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \vec{x} & \longmapsto & \lambda\vec{x}\end{array}$ for $\lambda>0$ (also for $\begin{array}{ccc}\mathbb{R}^{p,q} & \longrightarrow & \mathbb{R}^{p,q} \\ \vec{x} & \longmapsto & \lambda\vec{x}\end{array}$)

  conf. factor $\Omega=\lambda^2$

- Stereographic projection



$$\begin{array}{ccc}\mathbb{R}^{n+1}\supset S^n\setminus\left\{\begin{smallmatrix}\text{north}\\ \text{pole}\end{smallmatrix}\right\} & & (x^0,x^1,\dots,x^n)\ ,\ \sum_{i=0}^{n}(x^i)^2=1 \\ \Big\downarrow \varphi & & \Big\downarrow \\ \mathbb{R}^n & & \frac{1}{1-x^0}(x^1,\dots,x^n)\end{array}$$

Exercise: show that φ is conformal with $\boxed{\Omega=\frac{1}{(1-x^0)^2}}$

- Any diffeomorphism $\varphi:\mathbb{R}^1\longrightarrow\mathbb{R}^1$, $\Omega=\left(\frac{d\varphi}{dx}\right)^2$

  equipped with metric $g=(dx)^2$

- Any (bi-) holomorphic map $\varphi:\begin{array}{ccc}D & \xrightarrow{\sim} & D' \\ \cap & & \cap \\ \mathbb{C} & & \mathbb{C}\end{array}$ (equipped with standard complex structure and Euclidean metric $g=(dx)^2+(dy)^2=\frac{1}{2}(dz\otimes d\bar{z}+d\bar{z}\otimes dz)$)

  $$\Omega=\left|\frac{\partial\varphi}{\partial z}\right|^2$$

Note: an (invertible) anti-holomorphic map $\varphi: D\xrightarrow{\sim}D'$ is an orientation-reversing conformal map with $\Omega=\left|\frac{\partial\varphi}{\partial\bar{z}}\right|^2$

- Möbius transformations $PSL_2(\mathbb{C})\circlearrowright\mathbb{CP}^1$

  $$\begin{pmatrix}a & b\\ c & d\end{pmatrix}: z\longmapsto\frac{az+b}{cz+d}=z'\ ,\quad \Omega=\left|\frac{\partial z'}{\partial z}\right|^2=\frac{1}{|cz+d|^4}$$

  with $ad-bc=1$

$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ translation by $b\in\mathbb{C}$, $\begin{pmatrix}e^{i\varphi/2} & 0\\ 0 & e^{-i\varphi/2}\end{pmatrix}$ rotation by angle $\varphi\in\mathbb{R}/2\pi\mathbb{Z}$, $\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}$ dilatation (rescaling) by factor $\lambda\in\mathbb{R}_+$, $\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}$ "special conformal transf." - will discuss later

Infinitesimal version of conformal maps

<u>def</u> A conformal vector field (also, "conformal Killing vector") on a (pseudo-)Riemannian mfd. $(\mathcal{M}, g)$ is a v.f. $v \in \mathrm{Vect}(\mathcal{M})$ satisfying

$$\underset{\substack{\uparrow \\ \text{Lie derivative of metric along } v}}{\mathcal{L}_v g} = \omega \cdot g \quad , \qquad \omega \in C^\infty(\mathcal{M}) \text{ is the infinitesimal conformal factor}$$

<u>Properties:</u>
- if u, v are two conf. v.f. with conf. factors ω_u, ω_v then
  $u+v$ is a conf. v.f. with $\omega_{u+v} = \omega_u + \omega_v$
  $[u,v]$ is a conf. v.f. with $\omega_{[u,v]} = \mathcal{L}_u \omega_v - \mathcal{L}_v \omega_u$
- Conf. v.fields comprise a Lie subalgebra $\mathrm{conf}(\mathcal{M}, g) \subset \mathrm{Vect}(\mathcal{M})$
- if $\mathcal{M}$ is compact, then $\mathrm{conf}(\mathcal{M}, g) = \mathrm{Lie}(\mathrm{Conf}(\mathcal{M}, g))$

with
$$\exp: \mathrm{conf} \longrightarrow \mathrm{Conf}$$
$$v \longmapsto \mathrm{Flow}_1(v) \qquad \text{(flow in unit time)}$$

Conformal vector fields on $\mathbb{R}^{p,q}$, $p+q>2$

In local coords $\{x^i\}$ on $(\mathcal{M}, g)$ the equation $\mathcal{L}_\varepsilon g = \omega \cdot g$ reads

(1) $\varepsilon^k \cdot \partial_k g_{ij} + (\partial_i \varepsilon^k) \cdot g_{kj} + (\partial_j \varepsilon^k) \cdot g_{ik} = \omega \cdot g_{ij}$

where $\varepsilon = \varepsilon^k(x) \partial_k$ is the v.f. in question, $g = g_{ij}(x)\, dx^i dx^j$ is the metric.

Specializing to $\mathbb{R}^{p,q}$ with $g_{ij} = \eta_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$ $\underbrace{\quad}_{p}\underbrace{\quad}_{q}$ equation (1) simplifies:

(2) $\boxed{\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega \cdot \eta_{ij}}$ where $\varepsilon_i = \eta_{ij} \varepsilon^j$ <we use η_{ij}, η^{ij} to raise and lower indices>

denote $n = p+q$

<u>Let us solve equation (2)</u>

- contracting (2) with η^{ij}, we get: $\boxed{\partial_i \varepsilon^i = \frac{n}{2} \omega}$ (3)
- applying ∂^j to (2): $\partial_i (\partial_j \varepsilon^j) + \partial_j \partial^j \varepsilon_i = \partial_i \omega \underset{(3)}{\Longrightarrow} \boxed{\partial_k \partial^k \varepsilon_i = (1 - \frac{n}{2}) \partial_i \omega}$ (4)
- applying ∂_j to (4): $\partial_k \partial^k \partial_j \varepsilon_i = (1-\frac{n}{2}) \partial_i \partial_j \omega$

$$\Big\Downarrow \text{ symm. } i \leftrightarrow j \text{, use (2)}$$

$$\boxed{\frac{1}{2} \eta_{ij} \partial_k \partial^k \omega = (1 - \frac{n}{2}) \partial_i \partial_j \omega} \qquad (5)$$

- $\partial^i (2)$: $\partial_k \partial^k (\partial_i \varepsilon^i) = (1-\frac{n}{2}) \partial_i \partial^i \omega \underset{(3)}{\Longrightarrow} \boxed{(n-1) \partial_i \partial^i \omega = 0}$ (6)
- (5), (6) imply that for $n \neq 1, 2$ $\boxed{\partial_i \partial_j \omega = 0}$ (7)

- Also, deriving (2) we get

(8) $\partial_i\partial_j \varepsilon_k + \partial_i\partial_k \varepsilon_j = \partial_i\omega \cdot \eta_{jk}$ $\quad i \leftrightarrow j$

Renaming indices:

(9) $\partial_j\partial_i \varepsilon_k + \partial_j\partial_k \varepsilon_i = \partial_j\omega \cdot \eta_{ik}$ $\quad j \leftrightarrow k$

(10) $\partial_k\partial_i \varepsilon_j + \partial_k\partial_j \varepsilon_i = \partial_k\omega \cdot \eta_{ij}$

(8) + (9) − (10) gives:

(11) $$\boxed{\partial_i\partial_j \varepsilon_k = \frac{1}{2}\left(\partial_i\omega \cdot \eta_{jk} + \partial_j\omega \cdot \eta_{ik} - \partial_k\omega \cdot \eta_{ij}\right)}$$

- Putting together (7) and (11), we get:

(12) $n \neq 1, 2 \Rightarrow \partial_i\partial_j\partial_k \varepsilon_\ell = 0$

I.e. ε^i depends on coordinates at most quadratically, also due to (7), ω is at most linear

General Ansatz: $$\begin{cases} \varepsilon_i(x) = a_i + b_{ij}x^j + c_{ijk}x^jx^k \\ \omega(x) = 2\mu + 4\nu_i x^i \end{cases}$$

(2) means:

- no restriction on a_i
- $b_{ij} + b_{ji} = 2\mu\,\eta_{ij} \Rightarrow b_{ij} = \underbrace{\beta_{ij}}_{\text{anti-symmetric in } i\leftrightarrow j} + \mu\,\eta_{ij}$
- $c_{ijk} + c_{jik} = 2\nu_k\,\eta_{ij} \underset{\text{like (11)}}{\Longrightarrow} c_{ijk} = \nu_j\eta_{ik} + \nu_k\eta_{ij} - \nu_i\eta_{jk}$

Therefore:

Liouville's thm: $\mathrm{conf}(\mathbb{R}^{p,q}) = \underbrace{\{\text{translations}\}}_{\simeq \mathbb{R}^n} \oplus \underbrace{\{\text{rotations}\}}_{\simeq SO(p,q)} \oplus \underbrace{\{\text{dilatations}\}}_{\simeq \mathbb{R}} \oplus \underbrace{\{\text{special conformal transformations}\}}_{\simeq \mathbb{R}^n}$

	conf. vector field $\varepsilon^i(x)$	infinitesimal conf. factor ω
translations	$\varepsilon^i(x) = a^i$	0
rotations	$\varepsilon^i(x) = \beta^i{}_j x^j$ (where $\beta_{ji} = -\beta_{ij}$)	0
dilatations	$\varepsilon^i(x) = \mu x^i$	2μ
special conformal transformations	$\varepsilon^i(x) = 2(x,\nu)\cdot x^i - \nu^i \lvert x\rvert^2$	$4(\nu, x)$

finite version:

	conf. map $\mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$	conf. factor Ω
translations	$x^i \longmapsto x^i + a^i \quad , \vec{a} \in \mathbb{R}^{p,q}$	1
rotations	$x^i \longmapsto O^i{}_j x^j$, where $O^i{}_j \in SO(p,q)$	1
dilatations	$x^i \longmapsto \lambda x^i \quad , \lambda \in \mathbb{R}_+$	λ^2
SCTs	$x^i \longmapsto \dfrac{x^i - \lvert x\rvert^2 b^i}{1-2(b,x)+\lvert b\rvert^2 \cdot \lvert x\rvert^2}$, where $\vec{b} \in \mathbb{R}^{p,q}$	$(1-2(b,x)+\lvert b\rvert^2 \cdot \lvert x\rvert^2)^{-2}$

Remarks:
- finite SCT can be written as $(x^i) \longmapsto (x'^i)$, where $\dfrac{x'^i}{\lvert x'\rvert^2} = \dfrac{x^i}{\lvert x\rvert^2} - b^i$

  i.e. an SCT is: (inversion) $\circ$ $\underset{\text{by } -\vec{b}}{(\text{translation})}$ $\circ$ (inversion)

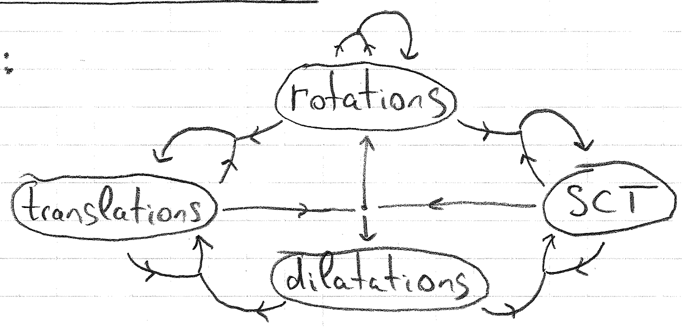
  where inversion is $\vec{x} \longmapsto \dfrac{\vec{x}}{\lvert x\rvert^2}$

- finite SCT is not everywhere well-defined as a map $\mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ since the denominator may vanish

  One can define the "conformal compactification" $N^{p,q} \supset \mathbb{R}^{p,q}$ so that SCTs are well-defined on $N^{p,q}$ (will discuss later)

Lie algebra structure on $\mathrm{conf}(\mathbb{R}^{p,q})$

structure constants (schematically):



Rem:
- conjugation by inversion is reflection $\leftrightarrow$ on this scheme - automorphism of $\mathrm{conf}(\mathbb{R}^{p,q})$
- there are several subalgebras, e.g. $\{\text{Poincaré}\} \oplus \{\text{dilatations}\}$

Thm: for $p+q > 2$, $\quad \mathrm{conf}(\mathbb{R}^{p,q}) \simeq so(p+1, q+1)$

- finite version: $\mathrm{Conf}_0(\mathbb{R}^{p,q}) \simeq SO_0(p+1, q+1)$ $\quad$ [or $SO_0(p+1,q+1)/\mathbb{Z}_2$ if -1 is in the connected comp. of 1]

(almost everywhere well-defined conf. maps) $\quad$ conn. comp. of 1

[for proof, cf. M. Schottenloher, "A mathematical introduction to CFT"]

Dimension counting

$$\mathrm{conf}(\mathbb{R}^{p,q}) = \{\text{translations}\} \oplus \{SO(p,q)\text{-rotations}\} \oplus \{\text{dilatations}\} \oplus \{SCT\}$$

dimensions: $\quad \dfrac{(n+1)(n+2)}{2} \quad = \quad n \quad + \quad \dfrac{n(n-1)}{2} \quad + \quad 1 \quad + \quad n$

$\quad \overset{\shortparallel}{\dim(so(p+1,q+1))}$

Action of $SO(p+1,q+1)$ on $\mathbb{R}^{p,q}$

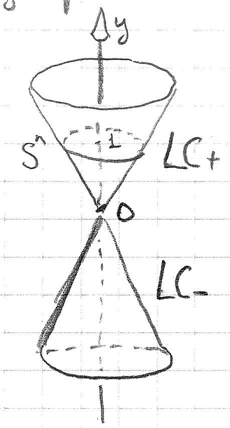
- Consider the case of $\mathbb{R}^n$, i.e. $p=n, q=0$

$SO(n+1,1)$ acts on $\mathbb{R}^{n+1,1}$ by linear isometries and preserves the light-cone

$$LC = \{(x^0,\dots,x^n,y) \in \mathbb{R}^{n+1,1} \mid (x^0)^2+\dots+(x^n)^2 - y^2 = 0\} \subset \mathbb{R}^{n+1,1}$$

"orthochronous Lorenz group" $SO_+(n+1,1)$ preserves the positive light-cone

$$LC_+ = \left\{(x^0,\dots,x^n,y) \in \mathbb{R}^{n+1,1} \,\middle|\, \begin{array}{l}(x^0)^2+\dots(x^n)^2-y^2=0 \\ y>0\end{array}\right\} \subset LC$$



We have two commuting actions:

$$SO_+(n+1,1) \circlearrowright LC_+ \circlearrowleft \mathbb{R}_+$$

dilatations

i.e. $SO_+(n+1,1)$ acts on $LC_+/\mathbb{R}_+ \sim S^n$ - section of $\mathbb{R}^+$ action, $S^n \subset LC_+$ singled out by $y=1$

LC_+ inherits <u>degenerate</u> metric from $\mathbb{R}^{n+1,1}$ (with 1-dim. kernel)

$\rightsquigarrow LC_+/\mathbb{R}_+$ inherits non-deg. conformal structure (the kernel is exactly killed by quotienting over $\mathbb{R}_+$)

so, $SO_+(n+1,1)$ acts on S^n by conformal maps

then we use stereographic projection $\quad S^n \setminus \{\text{north pole } (1,0,\dots,0)\}$

$$\downarrow$$

$$\mathbb{R}^n$$

Along the way, we identified S^n as the <u>conformal compactification</u> of $\mathbb{R}^n$:

conf. vect. fields extend to S^n, finite conf. maps are everywhere well-defined on S^n

<u>General case</u> $(\mathbb{R}^{p,q})$

$SO(p+1,q+1)$ acts on $\mathbb{R}^{p+1,q+1}$ and preserves $LC = \left\{(x^0,\dots,x^p,y^0,\dots,y^q) \in \mathbb{R}^{p+1,q+1} \,\middle|\, \sum_{i=0}^{p}(x^i)^2 - \sum_{j=0}^{q}(y^j)^2 = 0\right\}$, $LC \subset \mathbb{R}^{p+1,q+1}$

$$\begin{array}{c} SO(p+1,q+1) \circlearrowright LC\setminus\{0\} \circlearrowleft \mathbb{R}_+ \\ \downarrow \pi \\ \mathbb{R}P^{n+1} \end{array}$$

Denote the image of the light-cone in $\mathbb{R}P^{n+1}$ by $\mathcal{N}^{p,q}$. $\mathcal{N}^{p,q}$ inherits conformal structure from $\mathbb{R}^{p+1,q+1}$ and $SO(p+1,q+1)$ acts on $\mathcal{N}^{p,q}$ by conformal maps.

$$\iota: \mathbb{R}^{p,q} \longrightarrow \mathcal{N}^{p,q}$$

- injective, $\operatorname{im}(\iota)$ is open-dense in $\mathcal{N}^{p,q}$

$$(x^1,\dots,x^p,y^1,\dots,y^q) \longmapsto \left(\frac{1-\sum_1^p(x^i)^2+\sum_1^q(y^j)^2}{2} : x^1:\dots:x^p : \frac{1+\sum_1^p(x^i)^2-\sum_1^q(y^j)^2}{2} : y^1:\dots:y^q\right)$$

- $\mathcal{N}^{p,q}$ is the conf. compactification of $\mathbb{R}^{p,q}$
- $S^p \times S^q \xrightarrow{\pi} \mathcal{N}^{p,q}$ is the double cover

Conformal symmetry of $\mathbb{R}^2$

equation for a conf. vect. field: $\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega \delta_{ij} \iff \begin{cases} \partial_x \varepsilon_x = \partial_y \varepsilon_y = \frac{1}{2}\omega \\ \partial_x \varepsilon_y = -\partial_y \varepsilon_x \end{cases}$

(notation: $x = x^1$, $y = x^2$) $\iff \varepsilon_x + i\varepsilon_y$ satisfies Cauchy-Riemann equations

$\iff$ the vector field $\varepsilon_i \partial_i$ is of the form $\varepsilon(z)\frac{\partial}{\partial z} + \bar{\varepsilon}(\bar{z})\frac{\partial}{\partial \bar{z}}$

holom. vect. field — conjugate anti-hol v.f.

notation: $z = x+iy$, $\bar{z} = x - iy$

$\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$

conf. factor: $\omega = \partial\varepsilon + \bar{\partial}\bar{\varepsilon}$

Thus: $\left\{\begin{matrix}\text{conformal vect. fields} \\ \text{on } \mathbb{R}^2 \sim \mathbb{C}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{holom. vect.} \\ \text{fields on } \mathbb{C}\end{matrix}\right\}$. Lie algebra homomorphism

$\varepsilon_x \partial_x + \varepsilon_y \partial_y \longmapsto (\varepsilon_x + i\varepsilon_y)\cdot \partial_z$

$(\mathrm{Re}\,\varepsilon(z))\partial_x + (\mathrm{Im}\,\varepsilon(z))\partial_y \longleftarrow \varepsilon(z)\partial_z$

Finite version:

A diffeo $\varphi: \mathcal{D} \xrightarrow{\sim} \mathcal{D}'$, $\mathcal{D} \subset \mathbb{C}$, $\mathcal{D}' \subset \mathbb{C}$, is conformal iff φ is either holomorphic or anti-holomorphic

Proof: $\varphi^* \underset{dx^i dx^i = dz\, d\bar{z}}{g} = \frac{\partial\varphi^i}{\partial x^j}\frac{\partial\varphi^i}{\partial x^k} dx^j dx^k \underset{\substack{\text{setting } \varphi := \varphi^1 + i\varphi^2 \\ \bar{\varphi} = \varphi^1 - i\varphi^2}}{=} \frac{\partial\varphi}{\partial z}\cdot\frac{\partial\bar{\varphi}}{\partial z}(dz)^2 + \frac{\partial\varphi}{\partial z}\cdot\frac{\partial\bar{\varphi}}{\partial \bar{z}}\cdot dz\, d\bar{z} + \frac{\partial\varphi}{\partial\bar{z}}\cdot\frac{\partial\bar{\varphi}}{\partial z}\, d\bar{z}\cdot dz + \frac{\partial\varphi}{\partial\bar{z}}\cdot\frac{\partial\bar{\varphi}}{\partial\bar{z}}(d\bar{z})^2$

therefore

$\varphi^* g = \Omega g \iff$ either $\bar{\partial}\varphi = 0$, then $\Omega = |\partial\varphi|^2$
or $\partial\varphi = 0$, then $\Omega = |\bar{\partial}\varphi|^2$ $\square$

- $\mathrm{conf}(\mathbb{C}\setminus\{0\}) = \left\{\begin{matrix}\text{real parts of merom. vect. f. on } \mathbb{C} \\ \text{with pole at 0 allowed}\end{matrix}\right\}$

Introduce the Witt algebra $\mathcal{A} = \left\{ \sum_{n=-\infty}^{\infty} c_n \ell_n \,\middle|\, \begin{matrix} c_n \in \mathbb{C} \\ \ell_n = -z^{n+1}\frac{\partial}{\partial z}\end{matrix} \right\} = \left\{\begin{matrix}\text{merom. v.f. on } \mathbb{C} \\ \text{with pole at 0 allowed}\end{matrix}\right\}$

Generators $\{\ell_n\}$ satisfy commutation relations

$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n}$

$\mathrm{conf}(\mathbb{C}\setminus\{0\}) \hookrightarrow \underbrace{\overset{\text{merom. v.f.}}{\mathcal{A}}}_{\mathrm{Span}_{\mathbb{C}}\{\ell_n\}} \oplus \underbrace{\overset{\text{anti-merom. v.f}}{\bar{\mathcal{A}}}}_{\mathrm{Span}_{\mathbb{C}}\{\bar{\ell}_n\}}$

$\| $

$\mathrm{Span}_{\mathbb{R}}\{\ell_n + \bar{\ell}_n,\ i(\ell_n - \bar{\ell}_n)\}_{n=-\infty}^{\infty}$

$\ell_n = -z^{n+1}\partial_z$, $\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$

$$\begin{cases} [\ell_m, \ell_n] = (m-n)\,\ell_{m+n} \\ [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\,\bar{\ell}_{m+n} \\ [\ell_m, \bar{\ell}_n] = 0 \end{cases}$$

Exercise: show that $\mathcal{A}$ is the complexification of the Lie algebra of vector fields on a circle

- $\mathrm{conf}(\mathbb{C}) = \mathrm{Span}_{\mathbb{R}} \{ \ell_n + \bar{\ell}_n ,\ i(\ell_n - \bar{\ell}_n) \}_{n \geq -1}$
- $\left\{ \begin{matrix} \text{conf. v.f. on } \mathbb{C} \\ \text{vanishing at } 0 \end{matrix} \right\} = \mathrm{Span}_{\mathbb{R}} \{ \cdots \}_{n \geq 0}$
- $\mathrm{conf}(\underset{\overline{\mathbb{C}}}{\underset{\shortparallel}{\mathbb{C}P^1}}) = \mathrm{Span} \{ \ell_n + \bar{\ell}_n ,\ i(\ell_n - \bar{\ell}_n) \}_{n=-1,0,1} \simeq \mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{so}(3,1)$

$$\left.\begin{aligned} -(\ell_{-1} + \bar{\ell}_{-1}) &= \partial_x \\ -i(\ell_{-1} - \bar{\ell}_{-1}) &= \partial_y \end{aligned}\right\} \text{translations} \qquad \begin{aligned} -(\ell_0 + \bar{\ell}_0) &= x\partial_x + y\partial_y \quad \text{- dilatation} \\ -i(\ell_0 - \bar{\ell}_0) &= -y\partial_x + x\partial_y \quad \text{- rotation} \end{aligned}$$

$$\left.\begin{aligned} -(\ell_1 + \bar{\ell}_1) &= (x^2 - y^2)\partial_x + 2xy\,\partial_y \\ -i(\ell_1 - \bar{\ell}_1) &= -2xy\,\partial_x + (x^2 - y^2)\partial_y \end{aligned}\right\} \begin{matrix} \text{special conformal} \\ \text{transformations} \end{matrix}$$

- $\mathrm{Conf}(\mathbb{C}P^1) = \underbrace{PSL_2(\mathbb{C})}_{\text{Möbius transformations}} \simeq SO_+(3,1)$

- Despite the fact that $\mathrm{conf}(\mathbb{C})$ is ∞-dimensional, global conf. automorphisms of $\overline{\mathbb{C}}$ comprise only a finite-dimensional group.