CFT Lecture 3 (09.03.11): Conformal symmetry (continued)

Action of $SO(p+1,q+1)$ on $\mathbb{R}^{p,q}$

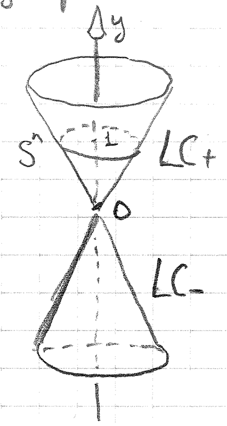
- Consider the case of $\mathbb{R}^n$, i.e. $p=n, q=0$

$SO(n+1,1)$ acts on $\mathbb{R}^{n+1,1}$ by linear isometries and preserves the light-cone

$$LC = \{(x^0,\dots,x^n,y) \in \mathbb{R}^{n+1,1} \mid (x^0)^2+\dots+(x^n)^2 - y^2 = 0\} \subset \mathbb{R}^{n+1,1}$$

"orthochronous Lorenz group" $SO_+(n+1,1)$ preserves the positive light-cone

$$LC_+ = \left\{(x^0,\dots,x^n,y) \in \mathbb{R}^{n+1,1} \,\middle|\, \begin{array}{l} (x^0)^2+\dots(x^n)^2-y^2=0 \\ y>0 \end{array}\right\} \subset LC$$



We have two commuting actions:

$$SO_+(n+1,1) \circlearrowright LC_+ \underset{}{\overset{\text{dilatations}}{\circlearrowleft}} \mathbb{R}_+$$

i.e. $SO_+(n+1,1)$ acts on $LC_+/\mathbb{R}_+ \sim \underset{\underset{LC_+}{\cap}}{S^n}$ - section of $\mathbb{R}^+$-action singled out by $y=1$

LC_+ inherits degenerate metric from $\mathbb{R}^{n+1,1}$ (with 1-dim. kernel)

$\rightsquigarrow LC_+/\mathbb{R}_+$ inherits non-deg. conformal structure (the kernel is exactly killed by quotienting over $\mathbb{R}_+$)

so, $SO_+(n+1,1)$ acts on S^n by conformal maps

then we use stereographic projection

$$\begin{tikzcd} S^n \setminus \{\text{north pole } (1,0,\dots,0)\} \arrow[d] \\ \mathbb{R}^n \end{tikzcd}$$

Along the way, we identified S^n as the conformal compactification of $\mathbb{R}^n$:

conf. vect. fields extend to S^n, finite conf. maps are everywhere well-defined on S^n

General case ($\mathbb{R}^{p,q}$)

$SO(p+1,q+1)$ acts on $\mathbb{R}^{p+1,q+1}$ and preserves $\underset{\mathbb{R}^{p+1,q+1}}{\overset{LC}{\cap}} = \left\{(x^0,\dots,x^p,y^0,\dots,y^q) \in \mathbb{R}^{p+1,q+1} \,\middle|\, \sum_{i=0}^{p}(x^i)^2 - \sum_{j=0}^{q}(y^j)^2 = 0\right\}$

$$\begin{tikzcd} SO(p+1,q+1) \circlearrowright LC\setminus\{0\} \circlearrowleft \mathbb{R}_+ \arrow[d, "\pi"] \\ \mathbb{R}P^{n+1} \end{tikzcd}$$

Denote the image of the light-cone in $\mathbb{R}P^{n+1}$ by $N^{p,q}$. $N^{p,q}$ inherits conformal structure from $\mathbb{R}^{p+1,q+1}$ and $SO(p+1,q+1)$ acts on $N^{p,q}$ by conformal maps.

$\iota: \mathbb{R}^{p,q} \longrightarrow N^{p,q}$ - injective, $\text{im}(\iota)$ is open-dense in $N^{p,q}$

$$(x^1,\dots,x^p,y^1,\dots,y^q) \longmapsto \left(\frac{1-\sum_1^p (x^i)^2+\sum_1^q (y^j)^2}{2} : x^1 : \dots : x^p : \frac{1+\sum_1^p (x^i)^2 - \sum_1^q (y^j)^2}{2} : y^1 : \dots : y^q\right)$$

- $N^{p,q}$ is the conf. compactification of $\mathbb{R}^{p,q}$
- $S^p \times S^q \xrightarrow{\pi} N^{p,q}$ is the double cover

$\boxed{3/2}$

Conformal symmetry of $\mathbb{R}^2$

equation for a conf. vect. field: $\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega \delta_{ij} \iff \begin{cases} \partial_x \varepsilon_x = \partial_y \varepsilon_y = \frac{1}{2}\omega \\ \partial_x \varepsilon_y = -\partial_y \varepsilon_x \end{cases}$

(notation: $\begin{matrix} x = x^1 \\ y = x^2 \end{matrix}$) $\iff \varepsilon_x + i\varepsilon_y$ satisfies Cauchy-Riemann equations

$\iff$ the vector field $\varepsilon_i \partial_i$ is of the form $\boxed{\underset{\substack{\uparrow \\ \text{holom. vect. field}}}{\varepsilon(z)\frac{\partial}{\partial z}} + \underset{\substack{\nwarrow \\ \text{conjugate} \\ \text{anti-hol v.f.}}}{\bar\varepsilon(\bar z)\frac{\partial}{\partial \bar z}}}$

notation: $z = x + iy$, $\bar z = x - iy$

$\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar\partial = \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$

conf. factor: $\boxed{\omega = \partial\varepsilon + \bar\partial\bar\varepsilon}$

Thus: $\left\{\begin{matrix}\text{conformal vect. fields} \\ \text{on } \mathbb{R}^2 \sim \mathbb{C}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{holom. vect.} \\ \text{fields on } \mathbb{C}\end{matrix}\right\}$. Lie algebra homomorphism

$$\varepsilon_x \partial_x + \varepsilon_y \partial_y \longmapsto (\varepsilon_x + i\varepsilon_y)\cdot\partial_z$$

$$(\mathrm{Re}\,\varepsilon(z))\partial_x + (\mathrm{Im}\,\varepsilon(z))\partial_y \longleftarrow\!\shortmid\ \varepsilon(z)\partial_z$$

Finite version:

A diffeo $\varphi: \underset{\underset{\mathbb{C}}{\cap}}{\mathcal{D}} \xrightarrow{\sim} \underset{\underset{\mathbb{C}}{\cap}}{\mathcal{D}'}$ is conformal iff φ is either holomorphic or anti-holomorphic

Proof: $\varphi^* \underbrace{g}_{dx^i dx^i = dz\, d\bar z} = \frac{\partial\varphi^i}{\partial x^j}\frac{\partial\varphi^i}{\partial x^k} dx^j dx^k \underset{\substack{\uparrow \\ \text{setting } \varphi := \varphi^1 + i\varphi^2 \\ \bar\varphi = \varphi^1 - i\varphi^2}}{=} \frac{\partial\varphi}{\partial z}\cdot\frac{\partial\bar\varphi}{\partial z}(dz)^2 + \frac{\partial\varphi}{\partial z}\cdot\frac{\partial\bar\varphi}{\partial\bar z}\cdot dz\, d\bar z + \frac{\partial\varphi}{\partial\bar z}\cdot\frac{\partial\bar\varphi}{\partial z}\, d\bar z\cdot dz + \frac{\partial\varphi}{\partial\bar z}\cdot\frac{\partial\bar\varphi}{\partial\bar z}(d\bar z)^2$

therefore

$\varphi^* g = \Omega g \iff$ either $\bar\partial\varphi = 0$, then $\Omega = |\partial\varphi|^2$
or $\partial\varphi = 0$, then $\Omega = |\bar\partial\varphi|^2$ $\square$

- $\mathrm{conf}(\mathbb{C}\setminus\{0\}) = \left\{\begin{matrix}\text{real parts of merom. vect. f. on } \mathbb{C} \\ \text{with pole at 0 allowed}\end{matrix}\right\}$

Introduce the Witt algebra $\mathcal{A} = \left\{ \sum_{n=-\infty}^{\infty} c_n \ell_n \,\middle|\, \begin{matrix} c_n \in \mathbb{C} \\ \ell_n = -z^{n+1}\frac{\partial}{\partial z}\end{matrix} \right\} = \left\{\begin{matrix}\text{merom. v.f. on } \mathbb{C} \\ \text{with pole at 0 allowed}\end{matrix}\right\}$

Generators $\{\ell_n\}$ satisfy commutation relations

$$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n}$$

$$\begin{tikzcd} \mathrm{conf}(\mathbb{C}\setminus\{0\}) \arrow[r, hook] \arrow[d, equal] & \underbrace{\overset{\substack{\text{merom. v.f.} \\ \downarrow}}{\mathcal{A}}}_{\mathrm{Span}_{\mathbb{C}}\{\ell_n\}} \oplus \underbrace{\overset{\substack{\text{anti-merom. v.f} \\ \downarrow}}{\bar{\mathcal{A}}}}_{\mathrm{Span}_{\mathbb{C}}\{\bar\ell_n\}} \\ \mathrm{Span}_{\mathbb{R}}\{\ell_n + \bar\ell_n, i(\ell_n - \bar\ell_n)\}_{n=-\infty}^{\infty} & \end{tikzcd}$$

$\ell_n = -z^{n+1}\partial_z$, $\bar\ell_n = -\bar z^{n+1}\partial_{\bar z}$

$$\begin{cases} [\ell_m, \ell_n] = (m-n)\,\ell_{m+n} \\ [\bar\ell_m, \bar\ell_n] = (m-n)\,\bar\ell_{m+n} \\ [\ell_m, \bar\ell_n] = 0 \end{cases}$$

Exercise: show that $\mathcal{A}$ is the complexification of the Lie algebra of vector fields on a circle

3/3

- $\mathrm{conf}(\mathbb{C}) = \mathrm{Span}_{\mathbb{R}} \{ \ell_n + \bar{\ell}_n , i(\ell_n - \bar{\ell}_n) \}_{n \geq -1}$
- $\left\{ \begin{matrix} \text{conf. v.f. on } \mathbb{C} \\ \text{vanishing at } 0 \end{matrix} \right\} = \mathrm{Span}_{\mathbb{R}} \{ \cdots \}_{n \geq 0}$
- $\underset{\overset{\parallel}{\overline{\mathbb{C}}}}{\mathrm{conf}(\mathbb{CP}^1)} = \mathrm{Span}\{ \ell_n + \bar{\ell}_n , i(\ell_n - \bar{\ell}_n) \}_{n=-1,0,1} \simeq sl_2(\mathbb{C}) \simeq so(3,1)$

$\left.\begin{matrix} -(\ell_{-1} + \bar{\ell}_{-1}) = \partial_x \\ -i(\ell_{-1} - \bar{\ell}_{-1}) = \partial_y \end{matrix}\right\}$ translations $\quad$ $-(\ell_0 + \bar{\ell}_0) = x\partial_x + y\partial_y$ - dilatation; $-i(\ell_0 - \bar{\ell}_0) = -y\partial_x + x\partial_y$ - rotation

$\left.\begin{matrix} -(\ell_1 + \bar{\ell}_1) = (x^2 - y^2)\partial_x + 2xy\,\partial_y \\ -i(\ell_1 - \bar{\ell}_1) = -2xy\,\partial_x + (x^2 - y^2)\partial_y \end{matrix}\right\}$ special conformal transformations

- $\mathrm{Conf}(\mathbb{CP}^1) = \underbrace{PSL_2(\mathbb{C})}_{\text{Möbius transformations}} \simeq SO_+(3,1)$
  - Despite the fact that $\mathrm{conf}(\mathbb{C})$ is ∞-dimensional, global conf. automorphisms of $\overline{\mathbb{C}}$ comprise only a finite-dimensional group.
- $\mathbb{C}$ does not have a conf. compactification (in the sense of a compact mfd containing $\mathbb{C}$, to which any conf. vect. field on $\mathbb{C}$ can be extended)
- Riemann's mapping thm: any two simply-connected domains $D, D' \subset \mathbb{C}$ are conformally equivalent
- Naïvely, $\mathbb{C} \setminus \{0\}$, punctured disc $D \setminus \{0\} = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$ and annulus $\mathrm{Ann}_{r,R} = \{ z \in \mathbb{C} \mid r < |z| < R \}$ have the same Lie algebras of conf. vect. fields:

$$\mathrm{conf}(\mathbb{C} \setminus \{0\}) \simeq \mathrm{conf}(D \setminus \{0\}) \simeq \mathrm{conf}(\mathrm{Ann}_{r,R}) \simeq \mathcal{A} = \mathrm{Span}_{\mathbb{C}} \{ \ell_n \}_{-\infty}^{\infty}$$

But there is a subtlety: different convergence restrictions for coefficients $\{c_n\}$ in Laurent expansion $\varepsilon(z)\partial_z = \sum_{n=-\infty}^{\infty} c_n \ell_n$ for these domains

Conformal symmetry of $\mathbb{R}^1$ (trivial case)

Riem. metric $g = (dx)^2$ $\quad$ $\left\{ \begin{matrix} \text{conformal diffeo} \\ \varphi : \mathbb{R}^1 \to \mathbb{R}^1 \end{matrix} \right\} = \left\{ \begin{matrix} \text{all diffeo} \\ \varphi : \mathbb{R}^1 \to \mathbb{R}^1 \end{matrix} \right\}$ $\quad$ conf. factor: $\Omega = \left(\frac{\partial \varphi}{\partial x} \right)^2$

$\mathrm{conf}(\mathbb{R}^1) = \mathrm{Vect}(\mathbb{R}^1)$; vect. field $\varepsilon(x)\partial_x$ has conf. factor $\omega = 2\partial_x \varepsilon(x)$

$\underset{\overset{\parallel}{\overline{\mathbb{R}}^1}}{\mathrm{Conf}(S^1)} = \mathrm{Diff}(S^1) \supset \underbrace{PSL_2(\mathbb{R})}_{\text{Möbius transf. of } S^1} \simeq SO_+(2,1)$ - "restricted conformal group of $\mathbb{R}$"

Again: S^1 is not the true conf. compactification of $\mathbb{R}^1$, it is rather the "imposed" compactification

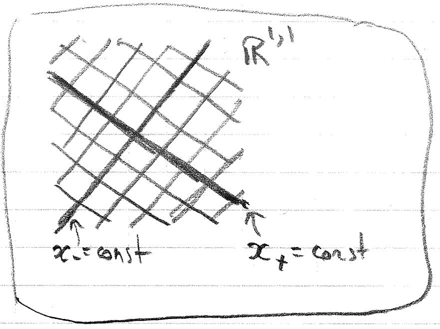
Conformal symmetry of Minkowski plane $\mathbb{R}^{1,1}$

Minkowski metric: $g = (dx)^2 - (dy)^2 = \eta_{ij} dx^i dx^j$ where $\eta_{ij} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

Switch to light-cone coordinates:

$\begin{cases} x_+ = x+y \\ x_- = x-y \end{cases}$ useful f-lae: $\begin{cases} x = \frac{x_+ + x_-}{2} \\ y = \frac{x_+ - x_-}{2} \end{cases}$ $\partial_+ := \frac{\partial}{\partial x_+} = \frac{\partial_x + \partial_y}{2}$ $\partial_x = \partial_+ + \partial_-$

$\partial_- := \frac{\partial}{\partial x_-} = \frac{\partial_x - \partial_y}{2}$ $\partial_y = \partial_+ - \partial_-$



$g = dx_+ dx_-$, $\eta_{ij} = \begin{pmatrix} & 1/2 \\ 1/2 & \end{pmatrix}$

Equation $\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega \eta_{ij}$ for components of a conf. vector field $\varepsilon^i \partial_i = \varepsilon_+(x_+, x_-)\partial_+ + \varepsilon_-(x_+, x_-)\partial_-$ reads:

$$\begin{cases} \partial_- \varepsilon_+ = 0 \\ \partial_+ \varepsilon_- = 0 \\ \partial_+ \varepsilon_+ + \partial_- \varepsilon_- = \omega \end{cases}$$

Therefore: a generic conf. v.f. on $\mathbb{R}^{1,1}$ is a vect. field of the form $\boxed{\varepsilon_+(x_+)\partial_+ + \varepsilon_-(x_-)\partial_-}$

it has conf. factor $\boxed{\omega = \partial_+ \varepsilon_+ + \partial_- \varepsilon_-}$

Thus $\mathrm{conf}(\mathbb{R}^{1,1}) \simeq \underbrace{\mathrm{Vect}(\mathbb{R})}_{\varepsilon_+ \partial_+} \oplus \underbrace{\mathrm{Vect}(\mathbb{R})}_{\varepsilon_- \partial_-}$

terminology:
functions of x_+ - "right-movers"
functions of x_- - "left-movers"

Now consider conf. maps

$$\varphi: \mathbb{R}^{1,1} \longrightarrow \mathbb{R}^{1,1}$$
$$(x_+, x_-) \longmapsto (\varphi_+, \varphi_-)$$

We have: $\varphi^* g = \varphi^*(dx_+ dx_-) = d\varphi_+ d\varphi_- =$

$$= \frac{\partial \varphi_+}{\partial x_+} \frac{\partial \varphi_-}{\partial x_+} (dx_+)^2 + \frac{\partial \varphi_+}{\partial x_+} \frac{\partial \varphi_-}{\partial x_-} dx_+ dx_- + \frac{\partial \varphi_+}{\partial x_-} \frac{\partial \varphi_-}{\partial x_+} dx_- dx_+ + \frac{\partial \varphi_+}{\partial x_-} \frac{\partial \varphi_-}{\partial x_-} (dx_-)^2$$

Hence:

φ is conformal $\Longleftrightarrow$ (1) either $\varphi_+ = \varphi_+(x_+), \varphi_- = \varphi_-(x_-)$

i.e. $\varphi \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ is a reparametrization of x_+ and x_- (independently)

(2) Or $\varphi_+ = \varphi_+(x_-), \varphi_- = \varphi_-(x_+)$

i.e. $\varphi = \begin{pmatrix} \text{reparam.} \\ \text{of } x_+, x_- \end{pmatrix} \circ \begin{pmatrix} \text{reflection} \\ (x,y) \mapsto (x,-y) \end{pmatrix}$

Conformal factors:

Case (1): $\Omega = (\partial_+ \varphi_+)(\partial_- \varphi_-)$ Case (2): $\Omega = (\partial_- \varphi_+)(\partial_+ \varphi_-)$

- $\mathrm{Conf}_0(\mathbb{R}^{1,1}) = \mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$ (the whole $\mathrm{Conf}(\mathbb{R}^{1,1})$ has 8 conn. components)
- "True" conf. compactification does not exist, but one may impose $\overline{\mathbb{R}^{1,1}} = \underset{x_+}{\underset{\uparrow}{S^1}} \times \underset{x_-}{\underset{\uparrow}{S^1}}$
- $\mathrm{Conf}_0(\overline{\mathbb{R}^{1,1}}) = \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \underset{\text{Möbius}_+ \times \text{Möbius}_-}{\underset{\uparrow}{\supset}} PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \simeq SO(2,2) \leftarrow$ "restricted conf. group"

Exercise: $\left.\begin{array}{l} \text{identify 2 translations,} \\ \text{1 Lorentz boost, 1 dilatation,} \\ \text{2 SCTs} \end{array}\right\}$ as $\text{Möb}_+ \times \text{Möb}_-$-transf: $(x_+, x_-) \mapsto \left(\frac{a_+ x_+ + b_+}{c_+ x_+ + c_+}, \frac{a_- x_- + b_-}{c_- x_- + d_-} \right)$

Moduli of conformal structures (Some remarks)

def: A (pseudo-)Riemannian manifold (M,g) of signature (p,q) is called conformally flat if one can choose coord. charts on M, so that in each chart

$$g = \Omega(x)\, \eta_{ij}\, dx^i dx^j \quad \text{with } \eta_{ij} = \underbrace{\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}}_{p \qquad q} \quad \text{and } \Omega(x) > 0$$

Rem: Being conformally flat is a local property

- For $\dim(M) = 1, 2$ all (pseudo-)Riemannian manifolds are conformally flat (in case $\dim(M)=1$, actually globally flat)

  Exercise: prove this

- For $\dim(M) > 2$ not every metric is conf. flat:

  in case $\dim M \geq 4$ metric g is conf. flat $\Leftrightarrow$ Weyl curvature tensor $W(g)$ vanishes

  $W(g)$: (0,4) tensor constructed out of Riem. curvature, Ricci tensor, scalar curvature and metric

  in case $\dim M = 3$:

  metric g is conf. flat $\Leftrightarrow$ Cotton tensor $C(g)$ vanishes

  $C(g)$: (0,3) tensor, constructed out of covar. derivatives of Ricci tensor, scalar curvature and metric

def Moduli space of conf. structures on a smooth mfd. M is

$$\mathcal{M}_M := \left\{ \begin{matrix} \text{conf. structures} \\ \text{on } M \end{matrix} \right\} \Big/ \mathrm{Diff}(M)$$

- action $\mathrm{Diff}(M) \circlearrowright \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } M \end{matrix} \right\}$ is not free and has stabilizer $\mathrm{Conf}(M,\gamma)$ for a conf. structure $\gamma = g/\sim$, i.e. we have the following picture:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Conf}(M,\gamma) \arrow[r] & \mathrm{Diff}(M) \circlearrowright \left\{ \begin{matrix} \text{conf. str. } \gamma \\ \text{on } M \end{matrix} \right\} \arrow[d, two heads] \\ & & \mathcal{M}_M \end{tikzcd}$$

- Or infinitesimally:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{conf}(M,\gamma) \arrow[r] & \mathrm{Vect}(M) \underset{\text{infinitesimal action}}{\circlearrowright} \left\{ \begin{matrix} \text{conf. str. } \gamma \\ \text{on } M \end{matrix} \right\} \arrow[d, two heads] \\ & & \mathcal{M}_M \end{tikzcd}$$

(i.e. $\mathrm{Vect}(M)$ determines an integrable distribution on $\left\{ \begin{matrix} \text{conf. str.} \\ \text{on } M \end{matrix} \right\}$ whose leaf space is $\mathcal{M}_M$)

Case dim(M) > 2

if $\dim(M) \geq 4$ Weyl curvature tensor $W(g)$ is invar. under Weyl transf. $g \mapsto g' = \Omega \cdot g$

$\Rightarrow W(g) = W(g/\sim)$

for $\dim(M) = 3$, Cotton tensor $C(g)$ is invar. under Weyl transf. $\Rightarrow C(g) = C(g/\sim)$

Therefore, for $\dim(M) > 2$ conf. structures γ on M have <u>local moduli</u> $W(\gamma)$, $C(\gamma)$

so $\dim \mathcal{M}_M = \infty$

Case dim(M) = 2, conf. structures of signature (2,0)

notation: $\Sigma_{g,n,m}$ - 2-dim smooth oriented mfd ; $\Sigma_{g,n} := \Sigma_{g,n,0}$

genus # of punctures # of boundary circles

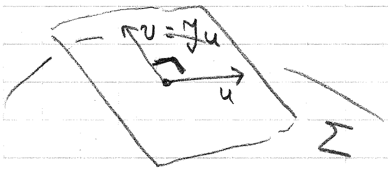
Rem: we only consider cx structures consistent with chosen orientation; and only (2,0)-conf. str.

- $\begin{Bmatrix} \text{conformal str.} \\ \text{on } \Sigma_{g,n,m} \end{Bmatrix} \xleftrightarrow[1 \text{ to } 1]{} \begin{Bmatrix} \text{complex str.} \\ \text{on } \Sigma_{g,n,m} \end{Bmatrix}$

<u>Reminder:</u> (almost) cx. str. on Σ is a section $J \in \Gamma(\Sigma, \mathrm{End}(T\Sigma))$ with $J_x^2 = -1$ for all $x \in \Sigma$

cx. str. = almost cx. + an integrability condition which holds automatically for $\dim \Sigma = 2$

$\longrightarrow$: given a conf. str. $\underset{g/\sim}{\overset{\shortparallel}{\gamma}}$ on Σ, we construct $J_x : T_x\Sigma \to T_x\Sigma$, $u \mapsto v$



where v is orthogonal to u and of the same length (w.r.t. g) and such that the pair (u, v) is positively oriented.

$\longleftarrow$: given a cx. str. J on Σ, choose arbitrary Riem. metric $\tilde{g}$ on Σ and deform it to

$g_x(u, v) := \tilde{g}(u, v')$ where v' is the projection of v along $J_x u$ to the line in $T_x\Sigma$ containing u

i.e. $v' = v - \alpha J_x u = \beta u$ for some $\alpha, \beta \in \mathbb{R}$

then take the conf. class $g/\sim$

- Terminology: $\Sigma_{g,n,m}$ endowed with conf. (or cx.) str. is called a <u>Riemann surface</u> (not a Riemannian mfd !)