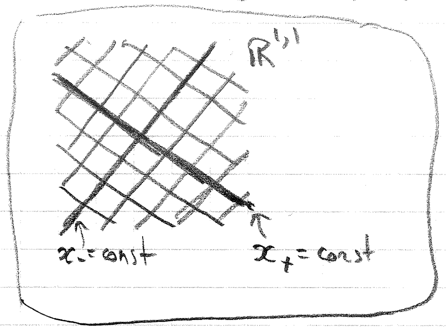
Conformal symmetry of Minkowski plane $\mathbb{R}^{1,1}$

Minkowski metric: $g = (dx)^2 - (dy)^2 = \eta_{ij}\,dx^i dx^j$ where $\eta_{ij} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

Switch to light-cone coordinates:

$\begin{cases} x_+ = x+y \\ x_- = x-y \end{cases}$ useful f-lae: $\begin{cases} x = \frac{x_+ + x_-}{2} \\ y = \frac{x_+ - x_-}{2} \end{cases}$ $\partial_+ := \frac{\partial}{\partial x_+} = \frac{\partial_x + \partial_y}{2}$ $\partial_- := \frac{\partial}{\partial x_-} = \frac{\partial_x - \partial_y}{2}$ $\partial_x = \partial_+ + \partial_-$ $\partial_y = \partial_+ - \partial_-$



$g = dx_+ dx_-$, $\eta_{ij} = \begin{pmatrix} & 1/2 \\ 1/2 & \end{pmatrix}$

Equation $\partial_i \varepsilon_j + \partial_j \varepsilon_i = \omega\,\eta_{ij}$ for components of a conf. vector field $\varepsilon^i \partial_i = \varepsilon_+(x_+, x_-)\partial_+ + \varepsilon_-(x_+, x_-)\partial_-$ reads:

$$\begin{cases} \partial_- \varepsilon_+ = 0 \\ \partial_+ \varepsilon_- = 0 \\ \partial_+ \varepsilon_+ + \partial_- \varepsilon_- = \omega \end{cases}$$

Therefore: a generic conf. v.f. on $\mathbb{R}^{1,1}$ is a vect. field of the form $\boxed{\varepsilon_+(x_+)\partial_+ + \varepsilon_-(x_-)\partial_-}$

it has conf. factor $\boxed{\omega = \partial_+ \varepsilon_+ + \partial_- \varepsilon_-}$

Thus $\mathrm{conf}(\mathbb{R}^{1,1}) \simeq \underbrace{\mathrm{Vect}(\mathbb{R})}_{\varepsilon_+\partial_+} \oplus \underbrace{\mathrm{Vect}(\mathbb{R})}_{\varepsilon_-\partial_-}$

terminology:
functions of x_+ – "right-movers"
functions of x_- – "left-movers"

Now consider conf. maps

$\varphi: \mathbb{R}^{1,1} \longrightarrow \mathbb{R}^{1,1}$

$(x_+, x_-) \longmapsto (\varphi_+, \varphi_-)$

We have: $\varphi^* g = \varphi^*(dx_+ dx_-) = d\varphi_+ d\varphi_- =$

$$= \frac{\partial\varphi_+}{\partial x_+}\frac{\partial\varphi_-}{\partial x_+}(dx_+)^2 + \frac{\partial\varphi_+}{\partial x_+}\frac{\partial\varphi_-}{\partial x_-}dx_+ dx_- + \frac{\partial\varphi_+}{\partial x_-}\frac{\partial\varphi_-}{\partial x_+}dx_- dx_+ + \frac{\partial\varphi_+}{\partial x_-}\frac{\partial\varphi_-}{\partial x_-}(dx_-)^2$$

Hence:

φ is conformal $\Longleftrightarrow$ (1) either $\varphi_+ = \varphi_+(x_+)$, $\varphi_- = \varphi_-(x_-)$

i.e. $\varphi \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$ is a reparametrization of x_+ and x_- (independently)

(2) Or $\varphi_+ = \varphi_+(x_-)$, $\varphi_- = \varphi_-(x_+)$

i.e. $\varphi = \begin{pmatrix} \text{reparam.} \\ \text{of } x_+, x_- \end{pmatrix} \circ \begin{pmatrix} \text{reflection} \\ (x,y) \mapsto (x,-y) \end{pmatrix}$

Conformal factors:

Case (1): $\Omega = (\partial_+\varphi_+)(\partial_-\varphi_-)$ Case (2): $\Omega = (\partial_-\varphi_+)(\partial_+\varphi_-)$

- $\mathrm{Conf}_0(\mathbb{R}^{1,1}) = \mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$ (the whole $\mathrm{Conf}(\mathbb{R}^{1,1})$ has 8 conn. components)
- "True" conf. compactification does not exist, but one may impose $\overline{\mathbb{R}^{1,1}} = \underset{x_+}{S^1} \times \underset{x_-}{S^1}$
- $\mathrm{Conf}_0(\overline{\mathbb{R}^{1,1}}) = \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \supset \underset{\text{Möbius}_+ \times \text{Möbius}_-}{PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})} \simeq SO(2,2)$ ← "restricted conf. group"

Exercise: identify 2 translations, 1 Lorentz boost, 1 dilatation, 2 SCTs $\Big\}$ as $\text{Möb}_+ \times \text{Möb}_-$-transf: $(x_+, x_-) \mapsto \left(\frac{a_+ x_+ + b_+}{c_+ x_+ + d_+}, \frac{a_- x_- + b_-}{c_- x_- + d_-} \right)$

Moduli of conformal structures (Some remarks)

<u>def</u>: A (pseudo-)Riemannian manifold (M,g) of signature (p,q) is called <u>conformally flat</u> if one can choose coord. charts on M, so that in each chart

$$g = \Omega(x)\, \eta_{ij}\, dx^i dx^j \quad \text{with } \eta_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix} \quad \text{and } \Omega(x) > 0$$

$$\underbrace{}_{p}\ \underbrace{}_{q}$$

<u>Rem</u>: Being conformally flat is a <u>local</u> property

- For $\dim(M) = 1, 2$ all (pseudo-)Riemannian manifolds are conformally flat (in case $\dim(M)=1$, actually globally flat)

  <u>Exercise</u>: prove this

- For $\dim(M) > 2$ not every metric is conf. flat:

  in case $\dim M \geq 4$ metric g is conf. flat $\Leftrightarrow$ Weyl curvature tensor $W(g)$ vanishes

  explicitly:
  $$W_{ijkl} = R_{ijkl} - \frac{2}{D-2}\left(g_{i[k}R_{l]j} - g_{j[k}R_{l]i}\right) + \frac{2}{(D-1)(D-2)}\, R \cdot g_{i[k}\, g_{l]j}$$

  $(0,4)$ tensor constructed out of Riem. curvature, Ricci tensor, scalar curvature and metric

in case $\dim M = 3$:

metric g is conf. flat $\Leftrightarrow$ Cotton tensor $C(g)$ vanishes

$(0,3)$ tensor, constructed out of cover. derivatives of Ricci tensor, scalar curvature and metric

explicitly:
$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{4}\left(\nabla_j R\, g_{ik} - \nabla_k R\, g_{ij}\right)$$

<u>def</u> Moduli space of conf. structures on a smooth mfd. M is

$$\mathcal{M}_M := \left\{ \begin{matrix} \text{conf. structures} \\ \text{on } M \end{matrix} \right\} \Big/ \mathrm{Diff}(M)$$

- action $\mathrm{Diff}(M) \circlearrowright \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } M \end{matrix} \right\}$ is <u>not free</u> and has stabilizer $\mathrm{Conf}(M, \gamma)$ for a conf. structure $\gamma = g/\sim$, i.e. we have the following picture:

$$0 \longrightarrow \mathrm{Conf}(M,\gamma) \longrightarrow \mathrm{Diff}(M) \circlearrowright \left\{ \begin{matrix} \text{conf. str. } \gamma \\ \text{on } M \end{matrix} \right\}$$
$$\big\downarrow$$
$$\mathcal{M}_M$$

- Or infinitesimally:

$$0 \longrightarrow \mathrm{conf}(M,\gamma) \longrightarrow \mathrm{Vect}(M) \underset{\text{infinitesimal action}}{\circlearrowright} \left\{ \begin{matrix} \text{conf. str. } \gamma \\ \text{on } M \end{matrix} \right\}$$
$$\big\downarrow$$
$$\mathcal{M}_M$$

(i.e. $\mathrm{Vect}(M)$ determines an integrable distribution on $\left\{ \begin{matrix} \text{conf. str.} \\ \text{on } M \end{matrix} \right\}$ whose leaf space is $\mathcal{M}_M$)

Case $\dim(M) > 2$

if $\dim(M) \geq 4$ Weyl curvature tensor $W(g)$ is invar. under Weyl transf. $g \mapsto g' = \Omega \cdot g$
$\Rightarrow W(g) = W(g/\sim)$

for $\dim(M) = 3$, Cotton tensor $C(g)$ is invar. under Weyl transf. $\Rightarrow C(g) = C(g/\sim)$

Therefore, for $\dim(M) > 2$ conf. structures γ on M have <u>local moduli</u> $W(\gamma)$, $C(\gamma)$

so $\dim \mathcal{M}_M = \infty$

Case $\dim(M) = 2$, conf. structures of signature $(2,0)$

notation: $\Sigma_{g,n,m}$ – 2-dim smooth oriented mfd ; $\Sigma_{g,n} := \Sigma_{g,n,0}$

g – genus, n – # of punctures, m – # of boundary circles

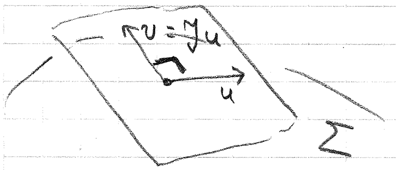
<u>Rem:</u> we only consider cx structures consistent with chosen orientation; and only $(2,0)$-conf. str.

- $\left\{\begin{matrix}\text{conformal str.} \\ \text{on } \Sigma_{g,n,m}\end{matrix}\right\} \xleftrightarrow[\text{1 to 1}]{} \left\{\begin{matrix}\text{complex str.} \\ \text{on } \Sigma_{g,n,m}\end{matrix}\right\}$

<u>Reminder:</u> (almost) cx. str. on Σ is a section $J \in \Gamma(\Sigma, \mathrm{End}(T\Sigma))$ with $J_x^2 = -1$ for all $x \in \Sigma$

cx. str. = almost cx. + an integrability condition which holds automatically for $\dim \Sigma = 2$

$\longrightarrow$: given a conf. str. $\gamma = g/\sim$ on Σ, we construct $J_x : T_x\Sigma \to T_x\Sigma$, $u \mapsto v$



where v is orthogonal to u and of the same length (w.r.t. g) and such that the pair (u,v) is positively oriented.

$\longleftarrow$: given a cx. str. J on Σ, choose arbitrary Riem. metric $\tilde{g}$ on Σ and deform it to

$g_x(u,v) := \tilde{g}(u,v')$ where v' is the projection of v along $J_x u$ to the line in $T_x\Sigma$ containing u

i.e. $v' = v - \alpha J_x u = \beta u$ for some $\alpha, \beta \in \mathbb{R}$

then take the conf. class $g/\sim$

- Terminology: $\Sigma_{g,n,m}$ endowed with conf. (or cx.) str. is called a <u>Riemann surface</u> (not a Riemannian mfd !)

- <u>Poincaré Uniformization Thm</u>

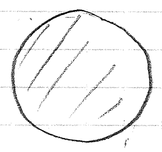
$\chi(\Sigma_{g,n,m}) = 2-2g-n-m < 0$ and $\Sigma_{g,n,m}$ is endowed with a $(2,0)$ conf. structure γ then there exists a unique choice of complete hyperbolic (i.e. $R \equiv -1$) metric g_{hyp} compatible with γ. Also, $(\Sigma_{g,n,m}, g_{hyp}) \underset{\text{isometric}}{\sim} \Pi_+ / \Gamma$

where $\Pi_+ = \{x+iy \mid y>0\}$ equipped with hyp. metric $\frac{1}{y^2}(dx^2+dy^2)$

and $\Gamma \subset PSL_2(\mathbb{R})$ is a discrete subgroup of the group of isometries of Π_+.

<u>Rem</u> ① <u>Hyperbolic Π_+</u> has other nice (isometric) models:

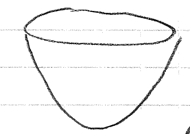
- <u>Poincaré disk</u>



$D \subset \mathbb{C}$, $|z|<1$ with hyp. metric $g_{hyp} = \frac{4\, dz\, d\bar{z}}{(1-|z|^2)^2}$

- <u>hyperboloid</u> $\{(x_1,x_2,x_3) \mid (x_1)^2+(x_2)^2-(x_3)^2 = -1,\ x_3>0\} \subset \mathbb{R}^{2,1}$

endowed with the metric coming from flat Minkowski metric on $\mathbb{R}^{2,1}$



② $\{\text{Isometries of } \Pi_+\} = \left\{\begin{matrix}\text{all conformal} \\ \text{autom. of } \Pi_+ \text{ (viewed as a disk)}\end{matrix}\right\} =$

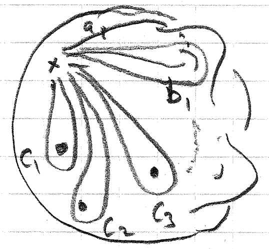
$= PSL_2(\mathbb{R}) \simeq SO_+(2,1)$

Möbius transformations of $\mathbb{C}$, preserving the real line $\mathbb{R} \subset \mathbb{C}$

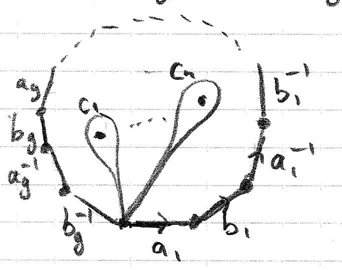
③ We automatically have $\underset{\substack{\cap \\ PSL_2(\mathbb{R})}}{\Gamma} \simeq \pi_1(\Sigma_{g,n,m})$ ← fundamental group

<u>Reminder</u>: $\pi_1(\Sigma_{g,n})$ can be given in terms of generators & relations:

$$\pi_1(\Sigma_{g,n}) = \langle a_1,\dots,a_g,b_1,\dots,b_g,c_1,\dots,c_n\rangle \Big/ (a_1b_1a_1^{-1}b_1^{-1})\cdots(a_gb_ga_g^{-1}b_g^{-1})c_1\cdots c_n = 1$$



or



$\Sigma_{g,n}$

Examples:
$$\pi_1(\Sigma_{0,n}) = \langle c_1,\dots,c_n\rangle / c_1\cdots c_n = 1 \simeq \langle c_1 \cdots c_{n-1}\rangle$$
$$\pi_1(\Sigma_{1,0}) = \langle a,b\rangle / aba^{-1}b^{-1} = 1 \simeq \mathbb{Z}^2$$

④ If $m \neq 0$ (boundary components are present), one can find another (unique) hyperbolic metric in a given conf. class: one for which boundary circles are geodesics (g^{gb}_{hyp})

<this metric is not complete; the covering is a complicated domain in Π_+>

⑤ Conversely, given a discrete subgroup $\Gamma \subset PSL_2(\mathbb{R})$, we recover a hyperbolic surface as $\Sigma := \Pi_+ / \Gamma$

Note: subgroup $x \Gamma x^{-1}$ (for any $x \in PSL_2(\mathbb{R})$) gives an isometric surface

⑥ Finally, we have 3 equivalent structures on $\Sigma_{g,n,m}$ (provided $\chi = 2-2g-n-m < 0$)

$$\begin{tikzcd}
\boxed{\text{Conformal structure } \gamma} \arrow[r, shift left] \arrow[d, shift right, "\text{Poincaré thm}"'] & \boxed{\text{Complex structure } J} \arrow[l, shift left] \\
\boxed{\text{Hyperbolic structure } g_{hyp}} \arrow[u, shift right, "\gamma = [g_{hyp}]"'] \arrow[r, leftrightarrow] & \left(\boxed{\begin{array}{c}\text{discrete subgroups } \Gamma \text{ of } PSL_2(\mathbb{R}) \\ \text{(modulo conjugation)}\end{array}}\right)
\end{tikzcd}$$

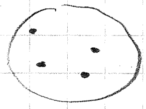
Mapping class group

(for $D=2$ also called "Teichmüller modular group")

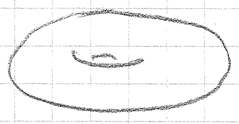
$MCG(\Sigma) := \pi_0(\mathrm{Diff}(\Sigma))$ · discrete group

We have: $0 \longrightarrow \underset{\substack{\uparrow \\ \text{connected comp. of } 1 \\ \text{in } \mathrm{Diff}(\Sigma)}}{\mathrm{Diff}_0(\Sigma)} \longrightarrow \mathrm{Diff}(\Sigma) \longrightarrow MCG(\Sigma) \to 0$

Examples: ① $MCG(\Sigma_{0,n})$ = "spherical braid group on n strands" $= \pi_1$ (<non-compactified> configuration space of n points on a sphere)



② $\underset{\substack{\text{torus} \\ = \mathbb{R}^2/\mathbb{Z}^2}}{MCG(\Sigma_{1,0})} = \underset{\substack{\text{(Möbius)} \\ \text{automorphisms} \\ \text{of } \mathbb{Z}^2 \text{ lattice}}}{PSL_2(\mathbb{Z})}$ ← the modular group



③ for a general $\Sigma_{g,n}$, $MCG(\Sigma)$ is given by generators (braid generators for punctures + Dehn twists along chosen cycles) + relations

- Mapping class group acts on $\pi_1(\Sigma)$ (by moving curves along diffeomorphisms)

  in fact, $MCG(\Sigma) = \underbrace{\mathrm{Aut}_{\text{outer}}}_{\mathrm{Aut}/\mathrm{Inn}}(\pi_1(\Sigma))$ ← "Dehn-Nielsen thm"

Teichmüller theory

idea: do the quotient $\mathcal{M}_{g,n} = \{\text{Conf. str. on } \Sigma_{g,n}\} / \text{Diff}(\Sigma_{g,n})$

in two steps:

define $T_{g,n} := \{\text{Conf. str. on } \Sigma_{g,n}\} / \text{Diff}_0(\Sigma_{g,n})$

then: $\mathcal{M}_{g,n} = T_{g,n} / MCG(\Sigma_{g,n})$

$T_{g,n}$ is called "Teichmüller space", it is a smooth mfd diffeo. to $\mathbb{R}^{6g-6+2n}$

it has a lot of structure:
- complex str. ⟵ Kähler str.
- several natural choices of metric, e.g. Weil-Petersson metric
- several natural coord. systems

action $MCG_{g,n} \circlearrowright T_{g,n}$ is not free, but has a discrete set of fixed points

$\mathcal{M}_{g,n} = T_{g,n} / MCG_{g,n}$ is an orbifold.

Rem: With boundary circles the story is the same, $T_{g,n,m} \sim \mathbb{R}^{6g-6+2n+3m}$, but there is no complex structure on $T_{g,n,m}$ and $\mathcal{M}_{g,n,m}$

How to describe the moduli of conf. structures?

1) Conf. structures as flat $PSL_2(\mathbb{R})$-connections

Reference: R.C. Penner "Decorated Teichmüller Theory"

Uniformization theorem gives a map

$$\left\{\begin{array}{c}\text{conf. structures}\\ \text{on } \Sigma_{g,n}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{subgroups } \Gamma \subset PSL_2(\mathbb{R})\\ \text{s.t. } \Gamma \simeq \pi_1(\Sigma_{g,n})\end{array}\right\} \Big/ PSL_2(\mathbb{R})$$

more specifically, we have:

$$T_{g,n} \xrightarrow{\ p\ } \text{Hom}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R})) / PSL_2(\mathbb{R})$$

$$\| $$

$$\left\{\begin{array}{c}\text{moduli space of flat } PSL_2(\mathbb{R})\text{-bundles}\\ \text{on } \Sigma_{g,n}\end{array}\right\}$$

Actually, p is injective and one can describe the image:

$$\text{im}(p) = \text{Hom}^{dfp}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R})) / PSL_2(\mathbb{R})$$

where: d - "discrete" (so that $1 \in PSL_2(\mathbb{R})$ is not an accumulation point of image of π_1)

f - "faithful" (injective)

p - "mapping peripherals to parabolics" (c_i; $\text{tr}\,\rho(c_i) = 2$)

Moduli space: $\mathcal{M}_{g,n} = MCG_{g,n} \backslash \text{Hom}^{dfp}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R})) / PSL_2(\mathbb{R})$

Coordinates on Hom^{dfp}: $\rho(a_1), \dots, \rho(a_g), \rho(b_1), \dots, \rho(b_g) \in PSL_2(\mathbb{R})$, $\rho(c_1), \dots, \rho(c_n) \in PSL_{parab}(\mathbb{R})$
subject to a relation.

(Exercise: compute the expected dimension of $\mathcal{M}_{g,n}$)