# Introduction to CFT

## Lecture 9 (04.05.11)

### Hamiltonian mechanics

data $\begin{cases} \text{"phase space" } (\Phi, \omega) \text{ - a symplectic mfd} & \text{- states} \\ \text{Hamitonian } H \in C^\infty(\Phi) & \text{- determines dynamics} \end{cases}$

$H \rightsquigarrow$ Ham. vect. field $\check{H} = \{H, \cdot\}$ s.t. $\iota_{\check{H}}\omega = -dH$

in Darboux coords $(p_i, q^i)$ on $\Phi$:
$\omega = \sum_i dp_i \wedge dq^i$
$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\right)$

time evolution = flow of $\check{H}$

i.e. $\boxed{\frac{d}{dt}Y^a(t) = \{H, Y^a(t)\}}$ for $\{Y^a\}$-coords on $\Phi$ and $[Y^a(t)]$ - a classical trajectory

- Hamilton's eq. of motion

Rem: equivalently, graph $[Y^a(t)] \subset \mathbb{R}\times\Phi$ is an integral trajectory for the vector field $\frac{\partial}{\partial t} + \check{H} \in \mathrm{Vect}(\mathbb{R}\times\Phi)$

(infinitesimal) Phase space symmetry: $\underset{\substack{\| \\ \{\psi,\cdot\}}}{\check{\psi}} \in \mathrm{Vect}(\Phi)$ s.t. $\underset{\substack{\| \\ \{H,\psi\}^\vee}}{[\check{H}, \check{\psi}]} = 0$ $\quad(\Leftrightarrow \{H,\psi\} = C \cdot \text{constant})$

- if $\{H,\psi\} = 0$ then $\check{H}\psi = 0 \Rightarrow \frac{d}{dt}\psi(Y(t)) = 0 \Rightarrow \psi$ is an integral of motion
- if $\{H,\psi\} = C$ then $\psi - Ct$ is an integral of motion

(Ex: particle in linear potential (= in constant force field) $H = \frac{p^2}{2m} + \varkappa \cdot X$, $\underset{\psi = p}{\check{\psi} = -\frac{\partial}{\partial x}}$, $p + \varkappa t$ - int. of motion)

Mixed symmetry: $\underset{\substack{\| \\ \{\psi,\cdot\}}}{\check{\psi}} - r(t)\frac{\partial}{\partial t} \in \mathrm{Vect}(\mathbb{R}\times\Phi)$ s.t. $\left[\frac{\partial}{\partial t} + \check{H}, -r(t)\frac{\partial}{\partial t} + \check{\psi}\right] = \underset{\substack{\| \\ -\dot{r}(t)}}{\boxed{f}} \cdot \left(\frac{\partial}{\partial t} + \check{H}\right)$

$\Leftrightarrow \{H,\psi\} + \dot{r}(t)H = C(t)$

$\Rightarrow \boxed{\psi + r(t)H - \int^t C(\tilde{t})\,d\tilde{t}}$ is an integral of motion

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## Lagrangian → Hamiltonian formalism

given $X, L \in C^\infty(TX)$, one sets $\Phi = \underbrace{T^*}_{p_i}\underbrace{X}_{x^i}$, $\omega = \sum_i dp_i \wedge dx^i = d\alpha$

$\alpha = \sum_i p_i\, dx^i$

assuming the non-deg. condition $\det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0$, one has fiberwise diffeo

$$\begin{tikzcd} \lambda: TX \arrow[rr, "\sim"] \arrow[dr] & & T^*X \arrow[dl] \\ & X & \end{tikzcd}$$

$$(x^i, v^i) \longmapsto \left(x^i, p_i = \frac{\partial L}{\partial v^i}\right)$$

one sets

$$H = (\lambda^{-1})^* \left(v^i \frac{\partial L}{\partial v^i} - L\right)$$ - Legendre transform

Global description:

$$\begin{tikzcd} (T^* \oplus T^* \oplus T)X \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & (T \oplus T^* \oplus T^*)X \arrow[d, phantom, "\wr"] \\ T^*(TX) \arrow[r, "\sim", "\text{natural iso}"'] & T^*(T^*X) \\ \text{graph}(dL) \arrow[u, hook] \arrow[r, "\sim"] \arrow[d] & \text{graph}(dH) \arrow[u, hook] \arrow[d] \\ TX \arrow[r, "\lambda", "\sim"'] & T^*X \end{tikzcd}$$

(sign!)

**Rem:**

$$\left(\text{target symmetries in Lagr. formalism}\right) \xhookrightarrow{i} \left(\text{phase space symmetries}\right)$$

$\text{im}(i) = \{$Ham. v.f. $\check{\psi} \in \text{Vect}(\Phi)$ preserving $\alpha$, i.e. $\mathcal{L}_{\check{\psi}}\alpha = 0\}$ (cotangent lifts of vect. fields on $X$)

for such v.f. $\boxed{\psi = \iota_{\check{\psi}}\alpha}$ - Noether's integral of motion

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## Classical (Lagrangian) field theory

$$\left(\text{D-dimensional pseudo-Riemannian (e.g.) manifold } (\Sigma, g)\right) \longmapsto \left(\text{space of fields } \mathcal{F}_\Sigma, \text{ action } S_{\Sigma,g}: \mathcal{F}_\Sigma \to \mathbb{R}\right)$$

"worldsheet" or "space-time"

e.g.
$\text{Maps}(\Sigma, X)$
$\Gamma(\Sigma, E_\Sigma)$ ← some tensor bundle
$\text{Conn}(\Sigma, P_\Sigma)$ ← principal bdle

One requires:

- **locality**: $$S[\Phi(x)] = \int \underbrace{\text{vol}_\Sigma}_{\sqrt{\det g}\prod_\mu dx^\mu} \mathcal{L}_g(\Phi^a(x), \partial_\mu \Phi^a(x))$$
- **covariance**: to diffeo $m: \Sigma \to \Sigma'$ there is an associated map $m^*: \mathcal{F}_{\Sigma'} \to \mathcal{F}_\Sigma$

Action satisfies $$\boxed{S_{\Sigma,g}[\Phi] = S_{\Sigma', (m^{-1})^* g}[(m^{-1})^*\Phi]}$$

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Variation of action (under variation of fields only)

$$\delta S = \int_\Sigma \sqrt{g}\, dx \left( \frac{\partial \mathcal{L}}{\partial \phi^a} \delta\phi^a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \delta \partial_\mu \phi^a(x) \right) \overset{\text{Stokes' thm}}{=}$$

$$= \int_\Sigma \sqrt{g}\, dx \left( \frac{\partial \mathcal{L}}{\partial \phi^a} \underbrace{- \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right)}_{\nabla_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)} = \mathrm{div}_{\sqrt{g}dx} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\partial_\mu\right)} \right) \delta\phi^a(x) + \underbrace{\int_{\partial\Sigma} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)} \delta\phi^a(x) \cdot \iota_{\partial_\mu}(\sqrt{g}\, dx)}_{\text{flux of vect.field } \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a\partial_\mu \text{ through } \partial\Sigma}$$

- covariant divergence (w.r.t. Levi-Civita connection)

Action principle
→ Euler-Lagrange equation:

$$\underset{\frac{\delta S}{\delta\phi^a(x)}}{\boxed{\frac{\partial \mathcal{L}}{\partial \phi^a} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)} = 0}}$$

- PDE determining classical dynamics

Examples

① Free massive scalar field

for Lorentzian case, $g := |\det(g_{\mu\nu})|$

Sign convention depends on choice of signature:
+ for (3,1)
− for (1,3)

$$\mathcal{F}_\Sigma = C^\infty(\Sigma), \quad S_{\Sigma,g}[\phi] = \int_\Sigma \sqrt{g}\, dx \left( \frac{1}{2} (g^{-1})^{\mu\nu} \partial_\mu\phi \cdot \partial_\nu\phi + \frac{m^2}{2}\phi^2 \right)$$

$$\left( = \int_\Sigma \frac{1}{2} d\phi \wedge *(d\phi) + \frac{m^2}{2} \phi\wedge *\phi \right)$$

E-L equation: $\Big(\underbrace{\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\cdot\partial^\mu}_{\Delta} - m^2\Big)\phi = 0$ - Klein-Gordon equation

$\Delta$ - covar. Laplacian

$\overset{\shortparallel}{*d*d}$

(for $\Sigma = \mathbb{R}^{3,1}$: $(\partial_\mu\partial^\mu - m^2)\phi = 0$

plane wave solutions: $e^{i(p,x)}$ with $(p,p) = -m^2$

Generic sol.: $\phi(x) = \int_{MS \subset \mathbb{R}^{3,1}} e^{i(p,x)} \rho(p)$ "mass shell" $(p,p) = -m^2$ )

①′ Non-free —″— : $S = \int_\Sigma \sqrt{g}\, dx \left( \frac{1}{2}(g^{-1})^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi + U(\phi) \right)$

↑ polynomial of deg ≥3

E-L eq.: $\Delta\phi - U'(\phi) = 0$

- non-linear PDE

② Electrodynamics without charges

$\mathcal{F}_\Sigma = \Omega^1(\Sigma)$ field: $A = A_\mu(x)\, dx^\mu$, action $S = \frac{1}{4}\int_\Sigma \sqrt{g}\, dx\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int_\Sigma F\wedge *F$

$\wr\wr$

{ Connections in trivial principal $\mathbb{R}$-bundle over $\Sigma$ }

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$F = dA = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \in \Omega^2(\Sigma)$

- curvature of connection A

E-L eq: $\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu\nu}) = 0 \iff \nabla_\mu F^{\mu\nu} = 0 \iff$

$$\left.\begin{aligned} d^*F &= 0 \\ dF &= 0 \end{aligned}\right\} \text{Maxwell's equations}$$

+(tautological) Bianchi identity

**Rem:** this theory has gauge symmetry $A \mapsto A + d\varphi$, $\forall\varphi \in \Omega^0(\Sigma)$

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② ' Electrodynamics with (background) charges

$$S_{\Sigma,g,j}[A] = \int_\Sigma \sqrt{g}\, dx \left(\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu\right) = \int_\Sigma \tfrac{1}{2} F\wedge *F + A\wedge *j$$

$j \in$ ~~Vect(Σ)~~ $\Omega^1(\Sigma)$ - part of (background) geom. data

E-L eq.: $\nabla_\mu F^{\mu\nu} = j^\nu$ (or $d^* F = j$) - Maxwell's eq. with charges

**Rem**
- unless $\boxed{d^* j = 0}$, there are no solutions to E-L
- if $d^* j = 0$, $A \mapsto A + d\varphi$ changes action by a boundary term independent of $A$:

$$\delta S' = \int_{\partial\Sigma} \varphi \wedge *j$$ ; so it is still a gauge symmetry

② '' "massive vector field"

$$S'_{\Sigma,g}[A] = \int_\Sigma \sqrt{g}\, dx \left(\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \tfrac{m^2}{2} A_\mu A^\mu\right) = \int_\Sigma \tfrac{1}{2} F\wedge *F + \tfrac{m^2}{2} A\wedge *A$$

E-L: $\nabla_\mu F^{\mu\nu} - m^2 A^\nu = 0$ (or $d^* F - m^2 A = 0$)

- no gauge symmetry

## Target (internal) symmetry

finite: $\Phi(x) \longmapsto F_\alpha(\Phi(x))$ , $F_\alpha : X \to X$ , for $\mathcal{F}_\Sigma = \mathrm{Maps}(\Sigma, X)$

infinitesimal: $\Phi^a(x) \longmapsto \Phi^a(x) + f^a(\Phi(x))$ , $f^a = \frac{d}{d\alpha}\Big|_{\alpha=0} F^a_\alpha$

$$\delta_f S_\Sigma \underset{\mathrm{mod\ E\text{-}L}}{\sim} \int_{\partial\Sigma} f^a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \iota_{\partial_\mu}(\sqrt{g}\, dx)$$

Assuming $\delta_f S_N = 0$ for any submersed $N \subset \Sigma$, we have $\nabla_\mu J^\mu_f(x) \underset{\mathrm{mod\ E\text{-}L}}{\sim} 0$ where $J^\mu_f = f^a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)}$

- Noether thm for internal sym. in Lagr. class. f.t.

Noether's conserved current associated to symmetry f

**Rem**: • for $\Sigma$ of the form $[t_0, t_1] \times M$ with $\partial M = \emptyset$,

one constructs a conserved charge out of $J$ by setting

$$I_f(t) = \begin{pmatrix} \text{flux of } J^\mu \partial_\mu \\ \text{through } \{t\}\times M \end{pmatrix} = \int_{\{t\}\times M} J^0_f \underbrace{\iota_{\partial_0}(\sqrt{g}\, dx)}_{\mathrm{vol}_{\{t\}\times M}}$$ . Then $\nabla_\mu J^\mu \underset{\text{E-L}}{\sim} 0 \Rightarrow \frac{d}{dt} I_f \underset{\text{E-L}}{\sim} 0$

- Generally, Lagr. field theory on $\Sigma = [t_0, t_1] \times M$ can be viewed as classical Lagr. mechanics with $X_{\mathrm{mech}} = \mathrm{Maps}(M, X)$, $L = \int_{\{t\}\times M} \mathrm{vol}_{\{t\}\times M} \cdot \mathcal{L}(\Phi(t,\vec{x}), \partial_\mu \Phi(t,\vec{x}))$

(typically $\infty$-dimensional)
(if $\dim \Sigma \geq 2$)

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Ex. Charged (complex) scalar field

(can put any $\mathcal{U}(|\Phi|^2)$ here)

$$\mathcal{F} = \mathrm{Maps}(\Sigma, \mathbb{C}) \,, \quad S = \int \sqrt{g}\, dx \left( \frac{1}{2} \partial_\mu \Phi \cdot \partial^\mu \bar{\Phi} + \frac{m^2}{2} |\Phi|^2 \right) \quad (\in \mathbb{R})$$

E-L eq. $\begin{cases} (\Delta - m^2)\Phi = 0 \\ (\Delta - m^2)\bar{\Phi} = 0 \end{cases}$ Symmetry: $F_\alpha : \begin{matrix} \Phi \mapsto e^{i\alpha}\Phi \\ \bar{\Phi} \mapsto e^{-i\alpha}\Phi \end{matrix}$

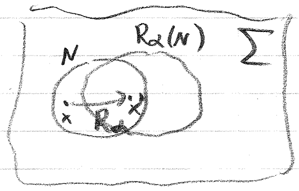
infinitesimally: $\left.\frac{d}{d\alpha}\right|_{\alpha=0} F_\alpha : \begin{matrix} \delta\Phi = i\Phi \\ \delta\bar{\Phi} = -i\bar{\Phi} \end{matrix}$

Noether current: $J^\mu = \frac{1}{2i} (\bar{\Phi}\partial^\mu \Phi - \Phi \partial^\mu \bar{\Phi}) = \mathrm{Im}(\bar{\Phi}\partial^\mu\Phi)$

conservation: $\nabla_\mu J^\mu = \frac{i}{2i}(\bar{\Phi}\Delta\Phi - \Phi\Delta\bar{\Phi}) \underset{\mathrm{mod}\, E\text{-}L}{\sim} \frac{1}{2i}(m^2|\Phi|^2 - m^2|\Phi|^2) = 0$

## Mixed (source-target) symmetry

$$\left.\begin{matrix} x' = R(x) \\ \Phi'(x') = F(\Phi(x)) \end{matrix}\right\} \quad \text{or} \quad \Phi(x) \longmapsto F_\alpha(\Phi(R_\alpha^{-1}(x)))$$

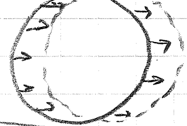


infinitesimally: $\Phi^a(x) \longmapsto \Phi^a(x) + \underbrace{f^a(\Phi(x)) - r^\mu(x)\partial_\mu \Phi(x)}_{Q \in \mathrm{Vect}(\mathcal{F}_\Sigma)}$

for any $N \underset{\text{submersed}}{\subset} \Sigma$, we have:

$$\underset{\left.\frac{d}{d\alpha}\right|_{\alpha=0} S_{R_\alpha(N)}[F_\alpha(\Phi(R_\alpha^{-1}(x)))]}{\underset{\shortparallel}{\delta_{f,r} S_N}} = \int_{\partial N} \mathcal{L}\, r^\mu \underbrace{(\mathrm{vol}_{\partial N})_\mu}_{\iota_{\partial_\mu}(\sqrt{g}\,dx)} + \underbrace{QS}_{\text{variation of } S \text{ due to variation } Q \text{ of fields only}} \overset{\mathrm{mod}\, E\text{-}L}{\sim} \int_{\partial N} \left( r^\mu \mathcal{L} - r^\nu \partial_\nu \Phi^a \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} f^a \right) \cdot (\mathrm{vol}_{\partial N})_\mu$$

boundary term coming from the shift of the integration domain by $r^\mu \partial_\mu$



$$\Rightarrow \boxed{\begin{matrix} \delta_{f,r} S_N = 0 \\ \text{for any submersed} \\ N \subset \Sigma \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \nabla_\mu J^\mu_{f,r}(x) \underset{\mathrm{mod}\, E\text{-}L}{\sim} 0 \\ \text{where } \boxed{J^\mu_{f,r} = r^\nu \left( \mathcal{L}\delta^\mu_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\nu \Phi^a \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} f^a} \end{matrix}}$$

## Examples

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- $\Sigma = \mathbb{R}^{p,q}$ (flat) $\rightsquigarrow$ source-translations $x^\mu \longmapsto x^\mu + a^\mu$ are symmetries (due to covariance, since transl. are isometries)

$$\mathcal{J}^\mu_{\underline{x} \mapsto \underline{x}+\underline{a}} = -T^\mu{}_\nu a^\nu$$

where $\boxed{T^\mu{}_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\nu \Phi^a - \mathcal{L}\,\delta^\mu_\nu}$ - the "canonical" stress-energy tensor

$$\partial_\mu T^\mu{}_\nu \underset{\text{mod } \mathcal{E}\text{-}L}{\sim} 0 \quad (\text{by Noether thm})$$

$$P_\mu(x^0) = \int_{\substack{x^0\text{-fixed} \\ \text{slice}}\subset\Sigma} T^0{}_\mu(x)$$ - Conserved energy-momentum (co-) vector $\longleftarrow$ i.e. $\frac{d}{dx^0} P_\mu(x^0) \underset{\text{mod } \mathcal{E}\text{-}L}{\sim} 0$

Ex: scalar field, $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\cdot\partial^\mu\Phi + \frac{m^2}{2}\Phi^2$,

$$T^\mu{}_\nu = \partial^\mu\Phi\cdot\partial_\nu\Phi - \delta^\mu_\nu\left(\tfrac{1}{2}\partial_\lambda\Phi\cdot\partial^\lambda\Phi + \tfrac{m^2}{2}\Phi^2\right)$$

- $\Sigma = \mathbb{R}^{p,q}$, source rotations (Lorentz transformations)

$$x^\mu \longmapsto x^\mu + \eta^{\mu\lambda}\omega_{\lambda\nu}x^\nu \longleftarrow \text{inf. rotation, } \omega_{\lambda\nu} = -\omega_{\nu\lambda}$$

$$\underset{\substack{\cap \\ so(p,q)}}{\omega} : \mathbb{R}^{p,q} \longrightarrow \mathbb{R}^{p,q}$$

for a scalar field, $\mathcal{J}^\mu_\omega(x) = -T^{\mu\nu}(x)\omega_{\nu\rho}x^\rho$, $\partial_\mu \mathcal{J}^\mu_\omega \underset{\text{mod } \mathcal{E}\text{-}L}{\sim} 0$

more generally (fields with spin), one can promote rotations of $\mathbb{R}^{p,q}$ to a mixed symmetry of action, with

$$x^\mu \longmapsto x^\mu + \omega^\mu{}_\nu x^\nu$$
$$\Phi^a \longmapsto \Phi^a + \boxed{f^a_\omega(\Phi)}$$

infinitesimal action of $so(p,q)$ on $X$

then $\mathcal{J}^\mu_\omega = -T^{\mu\nu}(x)\,\omega_{\nu\rho}x^\rho + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)} f^a_\omega$ is conserved

- $\Sigma = \mathbb{R}^{p,q}$, massless scalar field, dilatations: $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\cdot\partial^\mu\Phi$

$p+q=n$

$$\begin{cases} x^\mu \longmapsto e^\alpha x^\mu \\ \Phi \longmapsto e^{\frac{2-n}{2}\alpha}\,\Phi \end{cases}$$

$$\boxed{\mathcal{J}^\mu_{dil} = \frac{1}{2}\partial_\nu\Phi\cdot\partial^\nu\Phi\cdot x^\mu - \partial^\mu\Phi\cdot\partial_\nu\Phi\cdot x^\nu + \frac{2-n}{2}\partial^\mu\Phi\cdot\Phi}$$

## Hilbert stress-energy tensor:

$$\boxed{T^{\mu\nu} := -\frac{2}{\sqrt{g}}\frac{\delta S_g}{\delta g_{\mu\nu}}}$$ (works for any $(\Sigma, g)$!)

- symmetric, $T^{\mu\nu} = T^{\nu\mu}$
- conserved: $\nabla_\mu T^{\mu\nu} \underset{\text{mod } \mathcal{E}\text{-}L}{\sim} 0$ (due to covariance, $\int_N \frac{\delta S_g}{\delta g_{\mu\nu}}(\nabla_\mu r_\nu + \nabla_\nu r_\mu) \underset{\mathcal{E}\text{-}L}{\sim} \int_{\partial N}(\dots) \Rightarrow \int_N \nabla_\mu T^{\mu\nu} r_\nu \sim$ surface term $\Downarrow$ $\nabla_\mu T^{\mu\nu} \sim 0$)
- does <u>not</u> necessarily coincide with $T^{\mu\nu}_{canonical}$ on flat space-times