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MATHEMATICS

On Simplicial *BF* Theory[1]

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1. INTRODUCTION

This paper is a brief exposition of constructions and results from electronic preprint [4] obtained by the author in the framework of the simplicial program for topological quantum field theories. The idea of the simplicial program is an equivalent transfer of a field theory on a manifold M in Lagrangian formalism to a discretized (simplicial) theory associated with a triangulation T of M with some finite-dimensional space of fields. Equivalent transfer means that we must define the space of fields, action, and observables of the simplicial version of the theory, so that the correlation functions in this version coincide with those in the initial theory on the manifold. Such a transfer would allow us to calculate the partition function and correlation functions of quantum field theory by means of finite-dimensional integrals rather that functional ones, thus obviating the necessity to solve the divergence and renormalization problems. We consider non-Abelian BF theory with the classical action $S = \mathrm{tr}\int_M B \wedge (dA + A \wedge A)$, where the field A is a connection on the trivial principle G-bundle on M (by G we denote the gauge group) and B is a g-valued $(\dim M - 2)$-form on M ($\mathfrak{g}$ denotes the Lie algebra of G). We do not discuss the observables. The plan of work is as follows. First, the BF theory in Batalin–Vilkovisky formalism (for BF theory, this means replacing A and B by inhomogeneous differential forms) is interpreted as a special case of "abstract" BF theory, which corresponds to the differential graded Lie algebra of $\mathfrak{g}$-valued differential forms on M (this is an implementation of ideas from [1]). Then, for an abstract BF theory, we define an effective (induced) theory on a subcomplex by means of the BV integral. By construction, the induced action, like the initial one, satisfies the Batalin–Vilkovisky quantum master equation. Simplicial BF theory corresponds to induction from $\mathfrak{g}$-valued differential forms to $\mathfrak{g}$-valued cell cochains on the triangulation. The action of simplicial BF theory has the property of simplicial locality: it splits into the sum of local contributions of individual simplices of the triangulation. Hence, to know the action of simplicial BF theory, it suffices to perform a universal computation for one simplex in each dimension. This computation is performed explicitly in dimensions 0 and 1; for higher dimensions, perturbative results are obtained (the values of first few Feynman diagrams for the BV integral are computed).

The construction of inducing an effective theory by the BV integral can also be interpreted in terms of homotopy algebra: the action of abstract BF theory can be understood as a generating function for the operations of differential graded Lie algebra, while the action of the induced theory is understood as a generating function for the quantum L_∞ structure on a subcomplex, and the BV integral defines the homotopy transport. There also exist other constructions of induced infinity-algebras via the BV integral [6], [2].

Simplicial BF theory can be regarded as a tool providing topological invariants of manifolds and knots (after introducing observables into the theory) in terms of finite-dimensional integrals. To be more precise, starting from the simplicial BF theory on cell cochains of a triangulation, we can induce an effective action on the de Rham cohomology of the manifold. The tree part of the action on cohomology contains an information on the rational homotopy type of the manifold, while the quantum part provides some additional invariants of the manifold. The fact that there exist observables in BF theory associated to knots is known from [3]. The inclusion of simplicial versions of these observables into simplicial BF theory would allow us to compute some knot invariants in terms of finite-dimensional integrals.

[1] This article was translated by the author.

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2. ABSTRACT *BF* THEORY

Let V be a $\mathbb{Z}$-graded vector space. Let us construct a "space of fields" from V:

$$\mathcal{F} := V[-1] \oplus V^*[+2], \tag{1}$$

where $[k]$ denotes the shift in grading by k and V^* is the dual space for V. The ring of functions on $\mathcal{F}$ is defined as a symmetric algebra over $\mathcal{F}^*$ with coefficients in formal power series of $\hbar$:

$$\mathrm{Fan}(\mathcal{F}) := S^{\bullet}(V^*[+1] \oplus V[-2])[[\hbar]].$$

Here, $S^{\bullet}$ denotes the symmetric algebra and $\hbar$ is the Planck constant. We refer to the grading in V and V^* as the degree (deg) and to the grading in $\mathrm{Fun}(\mathcal{F})$ as the ghost number (gh). The canonical pairing of V and V^* defines a canonical Batalin–Vilkovisky structure on $\mathrm{Fun}(\mathcal{F})$ with canonical BV operator Δ. In turn, Δ defines a canonical odd Poisson bracket $\{\bullet, \bullet\}$ on $\mathrm{Fun}(\mathcal{F})$.

Let us also introduce two shifted identity maps ω: $V[-1] \to V$ and p: $V^*[+2] \to V^*$. We understand ω and p as two functions on $\mathcal{F}$ with values in V and V^*, respectively. We call ω and p fields, because they are generating functions for coordinate functions on $\mathcal{F}$.

Now, let V be equipped with the structure of a differential graded Lie algebra (DGLA), which means that two maps d: $V \to V$ and $[\bullet, \bullet]$: $V \otimes V \to V$ of degrees +1 and 0, respectively, are defined (these maps are the differential and the Lie bracket) so that the bracket is graded and skew-symmetric, and the three identities of DGLA hold: $d^2 = 0$, the Leibniz identity, and the Jacobi identity. We also require that V be unimodular, which means that, for each $\alpha \in V$, we have $\mathrm{Str}_V\,[\alpha, \bullet] = 0$ (here, Str_V is the supertrace over V). Given this DGLA structure, we the function

$$S := \left\langle p, d\omega + \frac{1}{2}[\omega, \omega]\right\rangle \in \mathrm{Fun}(\mathcal{F})$$

on V, which satisfies the Batalin–Vilkovisky quantum master equation (QME) $\Delta e^{S/\hbar} = 0$, or, equivalently, $\frac{1}{2}\{S, S\} + \hbar\Delta S = 0$. In the case under consideration, since S does not depend on $\hbar$, the QME splits into the classical and the quantum part: $\{S, S\} = 0$ and $\Delta S = 0$. The classical master equation in this case is equivalent to the three DGLA relations, namely, $d^2 = 0$, the Leibniz identity, and the Jacobi identity. The quantum part of the QME is equivalent to the unimodularity condition for V. The function S has zero ghost number $\mathrm{gh}(S) = 0$. We call S the action of abstract *BF* theory associated to V.

In coordinates, the construction is as follows. Let $\{e_i\}$ be a basis in V, and let $\{e^i\}$ be the dual basis in V^*; suppose also that $\{\omega^i\}$ and $\{p_i\}$ are the corresponding shifted bases in $V^*[+1]$ and $V[-2]$. We use the notation $|i| := \deg(e_i)$ for the degrees of basis elements in V. We have $\deg(e^i) = -|i|$, $\mathrm{gh}(\omega^i) = 1 - |i|$, and $\mathrm{gh}(p_i) = |i| - 2$. The algebra of functions $\mathrm{Fun}(\mathcal{F})$ is the algebra of power series in the variables $\{\omega^i\}$ and $\{p_i\}$ with coefficients in $\mathbb{R}[[\hbar]]$:

$$\mathrm{Fun}(\mathcal{F}) = \mathbb{R}[[\hbar]]\,[[\{\omega^i\}, \{p_i\}]].$$

The BV algebra structure is defined by the operator

$$\Delta := \sum_i \frac{\partial}{\partial\omega^i}\frac{\partial}{\partial p_i}$$

with ghost number $\mathrm{gh}(\omega) = +1$. The fields ω and p, which are understood as elements of the spaces $V^*[+1] \otimes V$ and $V^* \otimes V[-2]$, respectively, are $\omega = \sum_i \omega^i e_i$ and $p = \sum_i e^i p_i$.

Let d^i_j and f^i_{jk} be the structure constants of the differential and the Lie bracket in V, which means that $d(e_j) = \sum_i d^i_j e_i$ and $[e_j, e_k] = \sum_{j,k} f^i_{jk} e_i$. Then, the action is

$$S = \sum_{i,j} (-1)^{|j|+1} d^i_j p_i \omega^j + \frac{1}{2}\sum_{i,j,k} (-1)^{|j|(|k|+1)} f^i_{jk} p_i \omega^j \omega^k.$$

Thus, the action of abstract *BF* theory can be called the generating function for the structure constants of the DGLA operations on V. Moreover, as mentioned earlier, the quantum master equation generates the three relations in DGLA and the unimodularity condition.

The standard *BF* theory on a manifold M with gauge group G corresponds to the choice $V = \mathfrak{g} \otimes \Omega^{\bullet}(M)$.

3. EFFECTIVE ACTION

Suppose that V is a unimodular DGLA, V' is a deformation retract of V, and a pair of chain maps ι: $V' \to V$ and r: $V \to V'$ (an embedding and a retraction) such that $r\iota = \mathrm{id}_{V'}$ are given. Then, we have a decomposition of V into the sum of subcomplexes $V = \iota(V') \oplus V''$, where $V'' = \ker r$ and V'' is acyclic. Suppose also that K is a chain homotopy contracting V to V'. Namely, let K: $V \to V$ be a map of degree –1 such that $K\iota = 0$, $rK = 0$, $dK + Kd = 1 - \iota r$, and $K^2 = 0$. The triplet of maps (ι, r, K) defines the Hodge decomposition for V into the image of V', the d-exact part of V'', and the K-exact part of V'':

$$V = \iota(V') \oplus V''_{d-\mathrm{ex}} \oplus V''_{K-\mathrm{ex}}.$$

The maps d: $V''_{K-\mathrm{ex}} \to V''_{d-\mathrm{ex}}$ and K: $V''_{d-\mathrm{ex}} \to V''_{K-\mathrm{ex}}$ are the inverses of each other. The dual triplet (r^*, ι^*, K^*) defines the dual Hodge decomposition for V^*:

$$V^* = r^*(V'^*) \oplus V''^*_{d^*-\mathrm{ex}} \oplus V''^*_{K^*-\mathrm{ex}}.$$

For V' and V'', we construct the spaces of fields $\mathcal{F}' := V'[-1] \oplus V'^*[+2]$ and $\mathcal{F}'' := V''[-1] \oplus V''^*[+2]$ with the canonical BV structure. The fields ω and p are split as $\omega = \iota(\omega') + \omega''$ and $p = r^*(p') + p''$. Using the homotopy K, we construct a Lagrangian submanifold $\mathcal{L} := V''_{K-\mathrm{ex}}[-1] \oplus V''^{*}_{K^*-\mathrm{ex}}[+2] \subset \mathcal{F}''$.

Starting from the BF action S on $\mathcal{F}$, we construct the effective action $S' \in \mathrm{Fun}(\mathcal{F}')$ by means of an integral over $\mathcal{L}$ (the "BV integral"):

$$\exp\left(\frac{1}{\hbar}S'(\omega', p'; \hbar)\right) := \tag{2}$$

$$:= \int_{\mathcal{L}} \exp\left(\frac{1}{\hbar}S(\iota(\omega') + \omega'', r^*(p') + p'')\right)\mu_{\mathcal{L}}.$$

Here, $\mu_{\mathcal{L}}$ is the translation-invariant volume form on $\mathcal{L}$.

According to a theorem of Schwarz [7] (a version of Stokes' theorem for an integral of BV-coboundary over a Lagrangian submanifold), the effective action S', by construction, satisfies the quantum master equation on $\mathcal{F}'$: $\Delta' e^{S'/\hbar} = 0$. Here, Δ' denotes the canonical BV-operator on $\mathcal{F}'$. As usual in the BV formalism, the choice of a Lagrangian submanifold $\mathcal{L}$ in (2) means the choice of a gauge fixing. Our choice to construct $\mathcal{L}$ from a Hodge decomposition is a version of the Lorenz gauge. According to another theorem of Schwarz, another choice for $\mathcal{L}$ in the same cobordism class of Lagrangian submanifolds leads to an equivalent effective action, which means that $e^{S'/\hbar}$ changes by a BV-coboundary (Δ'-exact term).

Integral (2) can be calculated perturbatively: S' is the sum of connected oriented Feynman diagrams, where outer edges going in or out and only trivalent inner vertices with a special orientation (two incoming and one outgoing edge) are allowed. This implies the existence of graphs of two types: trees, namely, binary rooted trees (without planar structure), and one-loop graphs, namely, graphs with one oriented cycle and several trees plugged into the cycle. There are no graphs with more than one loop for (2). To the inner edges we assign the propagator $-K$, to the leaves (incoming outer edges) we assign $\iota(\omega')$, and to the root (the outgoing edge) we assign the operation $\langle r^*(p'), \bullet\rangle$; at the vertices, we put the Lie bracket $[\bullet, \bullet]$. The value of a graph contains also its symmetric coefficient and the factor $\hbar$ (if the graph has a loop).

The first terms of the perturbative expansion for (2) are

$$S'(\omega', p'; \hbar) = \langle p', d\omega'\rangle + \frac{1}{2}\langle p', r([\iota(\omega'), \iota(\omega')])\rangle$$

$$-\frac{1}{2}\langle p', r([K[\iota(\omega'), \iota(\omega')], \iota(\omega')])\rangle$$

$$+ \ldots - \hbar \mathrm{Str} K[\iota(\omega'), \bullet] + \hbar\frac{1}{2}\mathrm{Str} K[\iota(\omega'), K[\iota(\omega'), \bullet]]$$

$$+ \hbar\frac{1}{2}\mathrm{Str} K[K[\iota(\omega'), \iota(\omega')], \bullet] + \ldots$$

Here, the supertraces corresponding to one-loop graphs are calculated over $V[-1]$.

The general structure of effective BF theory is

$$S'(\omega', p'; \hbar) = \sum_i p'_i Q'^i(\omega') + \hbar \ln\rho'(\omega'). \tag{3}$$

Here, the first term is the contribution of trees and the second is that of 1-loop diagrams. It is convenient to construct a first-order differential operator from Q'^i: $Q' = Q'^i(\omega')\frac{\partial}{\partial\omega'^i}$ on $\mathrm{Fun}(V'[-1]$ (otherwise understood as a vector field on $V'[-1]$), the BRST differential of effective theory. Thus, the action S' is constructed from a vector field Q' and a function ρ' on $V'[-1]$. The quantum master equation for S' is equivalent to the pair of relations $Q'^2 = 0$ (the classical part of the QME) and $\rho' \mathrm{div} Q' + Q'(\rho') = 0$ (the quantum part). The first relation means that Q' defines a cohomological vector field on $V'[-1]$, and the second means that ρ' is the density of a Q'-invariant measure on $V'[-1]$: $\mathrm{Lie}_{Q'}(\rho'\mu_{V'[-1]}) = 0$), where $\mathrm{Lie}_{Q'}$ is the Lie derivative along Q' and $\mu_{V'[-1]}$ is the translation-invariant measure on $V'[-1]$.

The effective BF action has a natural interpretation in terms of homotopy algebra: the tree (classical) part of the effective action is the generating function for the structure of an L_∞ algebra on V', and the sum over Feynman diagrams is equivalent to the classical formulas in the homotopy algebra for the induced L_∞ structure on a deformation retract. At the same time, the 1-loop part S' gives, in addition to this structure, the density of a Q'-invariant measure ρ' on $V'[-1]$. The triplet ($\mathrm{Fun}(V'[-1])$, Q', ρ') may be understood as a quantum version of the L_∞ structure on V'.

It is natural to extend the class of BF theories associated to unimodular DGLAs to the class of BF_∞ theories associated to quantum L_∞ algebras with action (3) and the canonical space of fields (1). The class of BF_∞ theories is closed under the operation of inducing an effective theory on a subcomplex.

4. SIMPLICIAL *BF* THEORY

Consider the following special case of the construction of an effective action. Let M be a manifold with triangulation T, and let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Suppose that $V = \mathfrak{g} \otimes \Omega^\bullet(M)$ is the DGLA of $\mathfrak{g}$-valued differential forms on M, where the differential is the usual exterior derivative for forms and the Lie bracket is defined by wedge multiplication of forms and commutator in coefficients. As mentioned above, the abstract BF theory associated to such V is the usual BF theory. The deformation retract we are interested in is the complex of $\mathfrak{g}$-valued cell cochains on the triangulation T: $V' = \mathfrak{g} \oplus C^\bullet(T)$. We call the effective action S'

for this case the action of simplicial BF theory on the triangulation T. Let us denote this action by S_T.

Here, we use the following Hodge decomposition data (ι, r, K) defining the gauge fixing in the BV integral. For the embedding ι we use the Whitney map ι: $C^\bullet(T) \to \Omega^\bullet(M)$, which is an embedding of the complex of cell cochains on T into piecewise-linear differential forms on M. The retraction r: $\Omega^\bullet(M) \to C^\bullet(T)$ is defined by means of integrals over simplices of the triangulation. For the chain homotopy K: $\Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ we use the explicit construction due to Dupont [5] for the chain homotopy operator between the identity $\mathrm{id}_{\Omega^\bullet(M)}$ and the projection to Whitney forms ιr. Note that each of the three maps (ι, r, K) acts non-trivially on the forms $\Omega^\bullet(M)$ and trivially in the coefficients $\mathfrak{g}$.

The fields of simplicial BF theory are decomposed into basis cochains and chains as $\omega' = \sum_{\sigma \in T} \omega^\sigma e_\sigma$, $p' = \sum_{\sigma \in T} e^\sigma p_\sigma$, where $\omega^\sigma \in \mathfrak{g}[1 - |\sigma|]$ and $p_\sigma \in \mathfrak{g}^*[-2 + |\sigma|]$. Here, $|\sigma|$ denotes the dimension of the simplex σ. The action $S'(\omega', p'; \hbar)$ of simplicial BF theory possesses the following important property of "simplicial locality": it decomposes into a sum over all simplices σ of the triangulation of the terms $\bar{S}_\sigma(\{\omega^{\sigma'}\}_{\sigma' \subset \sigma}, p_\sigma; \hbar)$ depending only on the value of the field p at σ itself and on the values of ω at all faces of σ. In particular, this property means that, to obtain the action of simplicial BF theory on an arbitrary triangulation of an arbitrary manifold, it suffices to compute the simplicial action for one simplex in each dimension (with the standard triangulation). Thus, only one universal computation of integral (2) for a simplex in each dimension $D = 0, 1, 2, \ldots$ is sufficient.

In the dimension $D = 0$, the answer is trivial: for the 0-simplex, the induction from differential forms to cochains is the identity operation, since $V = V'$. The answer in this case is $\bar{S}_A = \left\langle p_A, \frac{1}{2}[\omega^A, \omega^A] \right\rangle_{\mathfrak{g}}$ (here, A is the name of the only vertex of the 0-simplex and $\langle \bullet, \bullet \rangle_{\mathfrak{g}}$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$).

In the dimension $D = 1$, the induction is nontrivial, but the BV integral can be calculated exactly, since it becomes the Gaussian for this case. The answer is

$$\bar{S}_{AB}$$
$$= \left\langle p_{AB}, \left[\frac{\omega^A + \omega^B}{2}, \omega^{AB}\right] - \left(\frac{\mathrm{ad}_{\omega^{AB}}}{2} \coth \frac{\mathrm{ad}_{\omega^{AB}}}{2}\right)(\omega^B - \omega^A) \right\rangle_{\mathfrak{g}}$$
$$+ \hbar \,\mathrm{tr}_{\mathfrak{g}} \ln \frac{\sinh \frac{\mathrm{ad}_{\omega^{AB}}}{2}}{\frac{\mathrm{ad}_{\omega^{AB}}}{2}}$$

(here, A and B are the vertices of the 1-simplex and $\mathrm{tr}_{\mathfrak{g}}$ is the trace over $\mathfrak{g}$).

In the higher dimensions $D \geq 2$, the BV integral is not Gaussian, and we cannot write an explicit formula for $\bar{S}_\sigma$ for a D-simplex σ. Nevertheless, we can calculate the values of individual Feynman diagrams for the BV integral and obtain a perturbative result, that is, first few terms of the power series in ω'. In particular, the contributions of tree diagrams with $c \leq 3$ leaves and loop diagrams with ≤ 2 leaves give the following perturbative approximation for $\bar{S}$ (up to $O(p'\omega'^4 + \hbar\omega'^3)$):

$$\bar{S}_\sigma = \left\langle p_\sigma, \sum_{\sigma_1 \subset \sigma} C^1_{\sigma, \sigma_1} \omega^{\sigma_1} + \frac{1}{2} \sum_{\sigma_1, \sigma_2 \subset \sigma} C^2_{\sigma, \sigma_1, \sigma_2} [\omega^{\sigma_1}, \omega^{\sigma_2}] \right.$$
$$\left. + \frac{1}{2} \sum_{\sigma_1, \sigma_2, \sigma_3 \subset \sigma} [[\omega^{\sigma_1}, \omega^{\sigma_2}], \omega^{\sigma_3}] \right\rangle_{\mathfrak{g}}$$
$$+ \frac{1}{2}\hbar \sum_{\sigma_1, \sigma_2 \subset \sigma} Q^2_{\sigma, \sigma_1, \sigma_2} \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_{\omega^{\sigma_1}} \mathrm{ad}_{\omega^{\sigma_2}}) + \ldots$$

Here, C^1, C^2, C^3, and Q^2 are combinatorial coefficients. Namely, $C^1_{\sigma, \sigma_1} = \pm 1$ if σ_1 is a face of σ of codimension 1 (the sign depends on the mutual orientation) and $C^1 = 0$ otherwise. Next, $C^2_{\sigma, \sigma_1, \sigma_2} = \pm \frac{|\sigma_1|!|\sigma_2|!}{(D+1)!}$ if σ_1 and σ_2 are a pair of faces of σ with |@[sigma]@1| + |@[sigma]@2| = @D and intersecting in only one vertex. Otherwise, $C^2 = 0$. The next coefficient is $C^3_{\sigma, \sigma^1, \sigma^2, \sigma^3} = \pm \frac{|\sigma_1|!|\sigma_2|!|\sigma_3|!}{(|\sigma_1| + |\sigma_2| + 1)(D+1)!}$ if σ_1, σ_2, and σ_3 are a triplet of faces of @ with |@[sigma]@1| + |@[sigma]@2| + |@[sigma]@3| = $D + 1$ and σ_1 and σ_2 have one common vertex, which belongs to σ_3. Otherwise, $C^3 = 0$. Finally, $Q^2_{\sigma, \sigma_1, \sigma_2} = \mathcal{A}_D + (D - 1)\mathcal{B}_D$ if σ_1 coincides with σ_2 and has dimension 1; $Q^2_{\sigma, \sigma_1, \sigma_2} = \pm 2\mathcal{B}_D$ if σ_1 and σ_2 have dimension 1 and intersect in one vertex; otherwise, $Q^2 = 0$. For the coefficient $\mathcal{A}_D$, we know an explicit expression in any dimension, which is $\mathcal{A}_D = \frac{(-1)^{D+1}}{(D+1)^2(D+2)}$. For $\mathcal{B}_D$, we know only the values for $D = 2, 3$; these are $\mathcal{B}_2 = \frac{1}{270}$ and $\mathcal{B}_3 = -\frac{1}{648}$.

Calculating of the coefficients C^1, C^2, and C^3 reduces to evaluating certain multiple integrals (since Dupont's construction gives an expression for the propagator K as a multiple integral). The problem of computing Q^2 is much more involved both technically and conceptually; solving it involves computing the supertrace of a certain integral operator on the space of differential forms on the simplex. This supertrace potentially contains a divergence (which is characteristic for the loop Feynman diagrams in quantum field theory). However, the explicit computations that we have carried out in the dimensions $D = 2, 3$ showed that this divergence is canceled for the first non-trivial quantum
1 operation (in the dimension 2, the cancelation of divergencies for all Feynman diagrams is observed).

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