relations: $\pi_* Q_M = Q_\Sigma$, $[Q_M^2 = 0]$,

$$\boxed{\iota_{Q_M}\omega_M = \delta S_M + \pi^*\alpha_\Sigma}$$

(cf. $\delta S_{cl} = \int_M EL\,\delta\varphi + \underbrace{\int_{\partial M}\cdots}_{\sim\alpha_\partial}$ in non-BV setting)

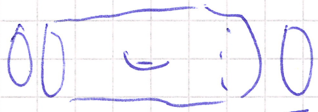
This implies:

- $L_{Q_M}\omega_M = \pi^*\omega_\Sigma$
- $L_{Q_M}(S_M) = \pi^*(-\iota_{Q_\Sigma}\alpha_\Sigma + 2S_\Sigma)$

  or $\boxed{\frac{1}{2}\iota_{Q_M}\iota_{Q_M}\omega_M = \pi^* S_\Sigma}$ $\leftarrow$ "mCME"

---

For a cobordism



$\Sigma_{in}$ $\quad$ $M$ $\quad$ $\Sigma_{out}$

$$\begin{tikzcd} & \mathcal{F}_M \arrow[dl, "\pi_{in}"'] \arrow[dr, "\pi_{out}"] & \\ \Phi_{\Sigma_{in}} & & \Phi_{\Sigma_{out}} \end{tikzcd}$$

gluing $\to$ (homotopy) fiber product

$$\begin{tikzcd} & \mathcal{F}_{M_I \cup_\Sigma M_{II}} \arrow[dl] \arrow[dr] & \\ \mathcal{F}_{M_I} \arrow[dr, "\pi_I"'] & & \mathcal{F}_{M_{II}} \arrow[dl, "\pi_{II}"] \\ & \Phi_\Sigma & \end{tikzcd}$$

Ex C-S $\quad M \mapsto \mathcal{F} = \Omega^\bullet(M,\mathfrak{g})[1]$, $\omega, Q, S$ - as before

$\Sigma \mapsto \Phi = \Omega^\bullet(\Sigma,\mathfrak{g})[1]$, $\omega_\Sigma = \int_\Sigma \frac{1}{2}\delta\mathcal{A}\wedge\delta\mathcal{A}$

$Q_\Sigma = \int_\Sigma \langle d\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}], \frac{\delta}{\delta\mathcal{A}}\rangle$ $\quad = \delta\underbrace{\int\frac{1}{2}\mathcal{A}\wedge\delta\mathcal{A}}_{\alpha_\Sigma}$

$S_\Sigma = \int_\Sigma \mathrm{tr}\,\frac{1}{2}\mathcal{A}\wedge d\mathcal{A} + \frac{1}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}$ $\quad = \int_\Sigma \mathrm{tr}\; c F_A + \frac{1}{2}A^+[c,c]$

$\mathcal{A}_\Sigma = \underset{1}{c} + \underset{0}{A} + \underset{-1}{A^+} \in \Omega^\bullet(\Sigma,\mathfrak{g})[1]$

compatibility with classical theory: $\quad \mathcal{F}^0 = F$, $S|_{\mathcal{F}^0} = S_{cl}$, $(\text{zero } Q)^0 = EL \subset F$

$\Phi^0 = \Phi_{cl}$, $\omega_\Sigma|_{\Phi^0} = \omega_\Sigma^{cl}$, $(\text{zero } Q_\Sigma)^0 = C \subset \Phi_{cl}$

Reduction: $M \longrightarrow \frac{\text{zero } Q_M}{Q_M} =: \mathcal{M}_M, \Pi_M \leftarrow$ +1-Poisson structure

$\qquad\qquad\qquad\qquad\qquad \downarrow \pi_*$ $\qquad\qquad$ • $\mathrm{im}\,\pi_* \subset \mathcal{M}_\Sigma$ Lagrangian

$\Sigma \longrightarrow \frac{\text{zero } Q_\Sigma}{Q_\Sigma} =: \mathcal{M}_\Sigma, \omega_\Sigma$ $\qquad$ • fibers of $\pi_*$ = symp. foliation of $\Pi_*$.

Examples: abelian C-S: $\quad M \to H^\bullet(M)[1]$

$\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \pi_* = \iota_\partial^*$

$\qquad\qquad\qquad\qquad \Sigma = \partial M \to H^\bullet(\Sigma)[1]$

---

Rem: we can always change a BV-BFV theory by a bdry term

$S_M \longmapsto S_M - \pi^* f_\Sigma$, $\quad \alpha_\Sigma \longmapsto \alpha_\Sigma + \delta f_\Sigma$, for $f_\Sigma \in C^\infty(\Phi_\Sigma)_0$

~~Quantization - idea~~

~~Fix $\Phi_\Sigma \downarrow \mathcal{B}$ a Lagrangian real, fibrating polarization~~

## Quantum BV-BFV theory - axiomatics

Data:

$(n-1)$-dim closed $\Sigma \longrightarrow (\mathcal{H}^\bullet_\Sigma, \Omega^\rho_\Sigma)$ a cochain complex (the BFV space of states)

$\rho \in R_\Sigma$ - poset of "realizations" of boundary theory

$n$-dim $M \longrightarrow$ (1) $(\mathcal{V}_r, \omega_r)$ - fin-dim -1-symplectic "space of residual fields"

$r \in R_M$ - poset of ~~admissible~~ "realizations" of the theory on $M$

$\downarrow R_\Sigma$

(2) $Z^r_M \in \mathcal{H}^\rho_\Sigma \otimes \mathrm{Dens}^{1/2}\mathcal{V}_r$ - partition function

satisfying $\left(\frac{i}{\hbar}\Omega^\rho_\Sigma - i\hbar\Delta_r\right) Z^r_M = 0$ - mQME

and defined mod $\left(\frac{i}{\hbar}\Omega^\rho_\Sigma - i\hbar\Delta_r\right)$-exact terms [due to ~~ambiguity~~ arbitrariness of gauge-fixing]

Axioms

- $\sqcup \longrightarrow \otimes$ for $\mathcal{H}$, $Z$; $\times$ for $\mathcal{V}_r$, $R_M$, $R_\Sigma$
- for $\rho' > \rho$, $(\mathcal{H}^{\rho'}, \Omega^{\rho'}) \rightsquigarrow (\mathcal{H}^\rho, \Omega^\rho)$, $P$ - surjective q-iso
- for $r' > r$ realizations, $P: \mathcal{V}_r \longrightarrow \mathcal{V}_{r'}$ -1-symplectic fibration,

$$Z^r_M = \underbrace{P_*}_{\text{BV pushforward}} Z^{r'}_M \quad [\text{if } \pi(r) = \pi(r'), \text{ otherwise: } Z^r_M = \rho P_* Z^{r'}_M ]$$

- for $M = M_I \cup_\Sigma M_{II}$, $Z^r_M = P_*\left(Z^{r_I}_{M_I} * Z^{r_{II}}_{M_{II}}\right)$, $*$ - pairing in $\mathcal{H}_\Sigma$

$R_M = R_{M_I} \times_{R_\Sigma} R_{M_{II}}$

for $\mathcal{V}_{r_I} \times \mathcal{V}_{r_{II}} \xrightarrow{P} \mathcal{V}_r$

---

## Quantization - idea

- fix $\Phi_\Sigma \xrightarrow{p} \mathcal{B}_\Sigma$ - Lagr. polarization (real, fibrating) s.t. $\alpha_\Sigma|_{\text{fiber of } p} = 0$

$$\pi^{-1}\{b\} = \mathcal{F}_b \subset \mathcal{F}_M \xrightarrow{\pi} \Phi_\partial \supset p^{-1}\{b\} \text{ (Lagr)}, \quad \Phi_\partial \xrightarrow{p} \mathcal{B}_\partial$$

$\mathcal{H}_\Sigma = \mathrm{Dens}^{1/2}\mathcal{B}_\Sigma$

$\Omega_\Sigma = \widehat{S}_\Sigma : \mathcal{H}^\bullet_\Sigma \to \mathcal{H}^{\bullet+1}_\Sigma$,
$\boxed{\alpha = p\,\delta b, \quad S_\Sigma(b,p) \rightsquigarrow \Omega_\Sigma = S_\Sigma\left(b, -i\hbar\frac{\partial}{\partial b}\right)}$ $\quad b \in \mathcal{B}_\partial$

- (i) $Z_M(b) = \int_{\mathcal{L} \subset \mathcal{F}_b} e^{\frac{i}{\hbar}S} \mu^{1/2} \in \mathrm{Dens}^{1/2}\mathcal{B}$ - usually ~~diverges~~ not perturbatively defined

$\mu^{1/2} = \mu^{1/2}_{\mathcal{B}} \cdot \mu^{1/2}_{\text{bulk}}$

- (ii) split $\mathcal{F}_b = \mathcal{V} \times \mathcal{F}_{\text{fluct}}$, $Z_M(b, \varphi) = \int_{\mathcal{L} \subset \mathcal{F}_{\text{fluct}}} e^{\frac{i}{\hbar}S(b + \varphi + \xi_{\text{fluct}})} \mu^{1/2} \in \mathrm{Dens}^{1/2}(\mathcal{B} \times \mathcal{V}) = \mathcal{H}_\partial \otimes \mathrm{Dens}^{1/2}(\mathcal{V})$, $\varphi \in \mathcal{V}$

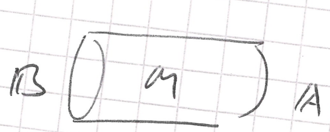
- $\mathcal{V}_{\text{res}} \xrightarrow{P} \mathcal{V}'_{\text{res}} \rightsquigarrow Z'_M = P_* Z_M$.

<u>Example</u>: abelian BF theory

$\mathcal{F} = \Omega^\bullet(M, E)[1] \oplus \Omega^\bullet(M, E^*)[n-2]$

$E$ ← $SL(n)$ - loc. system

$S = \int_M \langle B, dA\rangle + \int_{\partial_{in}} \langle B, A\rangle$ $\quad \omega = \int \langle \delta B, \delta A\rangle$




$\mathcal{F}_\partial = \Omega^\bullet(\partial M, E)[1] \oplus \Omega^\bullet(\partial M, E^*)[n-2]$

$\alpha_\partial = \int_{\partial_{out}} \langle B, \delta A\rangle - \int_{\partial_{in}} \langle \delta B, A\rangle$

$S_1 = \int_\partial \langle B, dA\rangle$

$= \int_\partial \langle B, \delta A\rangle - \delta \int_{\partial_{in}} \langle B, A\rangle$

A. Schwarz: for $M$ closed, $E$ acyclic, $Z = \tau(M, E) \in \mathbb{R}$

---

<u>Quantization</u>: $\mathbb{B} = \Omega^\bullet(\partial_{out}, E)[1] \oplus \Omega^\bullet(\partial_{in}, E^*)[n-2] \ni (\mathbb{A}, \mathbb{B})$

$\mathcal{H}_\partial = \mathrm{Dens}^{1/2}\mathbb{B} \ni \int_{\mathrm{Conf}_k(\partial_{in}) \times \mathrm{Conf}_\ell(\partial_{out})} \psi(x_1, \ldots x_k; y_1 \ldots y_\ell)\, \mathbb{B}(x_1) \ldots \mathbb{B}(x_k)\, \mathbb{A}(y_1) \ldots \mathbb{A}$

$\times \tau^{1/2}(\partial_{in})^{(-1)^{n-1}} \cdot \tau^{1/2}(\partial$

$\Omega_\partial = -i\hbar \left( \int_{\partial_{out}} d\mathbb{A} \frac{\delta}{\delta \mathbb{A}} + \int_{\partial_{in}} d\mathbb{B} \frac{\delta}{\delta \mathbb{B}} \right)$

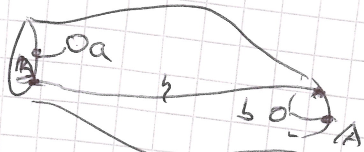
$\mathcal{V}_M = H^\bullet(M, \partial_{out})[1] \oplus H^\bullet(M, \partial_{in})[n-2] \ni (a, b)$

$Z(\mathbb{A}, \mathbb{B}; a, b; \hbar) = \xi \underbrace{\tau(M, \partial_{out}; E)}_{} \cdot \exp \frac{i}{\hbar}\left( \int_{\partial_{out}} b\mathbb{A} + \int_{\partial_{in}} \mathbb{B} a - \int_{\partial_{in} \times \partial_{out}} \mathbb{B}(x) \eta(x,y) \mathbb{A}(y) \right)$

$\mathrm{Det}\, H(M, \partial_{out}) / \pm 1 \cong \mathrm{Dens}^{1/2}_{\mathrm{const}} \mathcal{V}_M$

$\in \mathrm{Dens}^{1/2} \mathcal{V}_M \otimes \mathcal{H}_\partial$

$\eta \in \Omega^{n-1}(\mathrm{Conf}_2(M), E \boxtimes E^*)$ - propagator

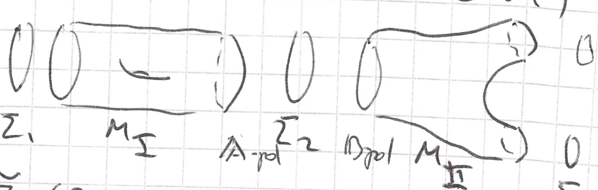



map $K: \alpha \mapsto \int_{M_2} \eta_{12}\, \alpha_2$

$\Omega(M, \partial_{out}; E) \to \Omega(M, \partial_{out}; E)$

is a chain homotopy btw id and projection to $H(M, \partial_{out}; E)$

- $Z$ satisfies mQME: $\left(\frac{i}{\hbar}\Omega_\partial - i\hbar \Delta_{res}\right) Z = 0$
- changing $\eta$ (and representatives of cohom) shifts $Z$ by $\left(\frac{i}{\hbar}\Omega_\partial - i\hbar\Delta_{res}\right)(\ldots)$
- gluing




(i) $\tilde{Z}_M(\mathbb{B}_1, \mathbb{A}_3; \underbrace{a_I, a_{II}, b_I, b_{II}}_{\tilde{\mathcal{V}}_M}) = \int Z_I\, e^{\frac{i}{\hbar}\int_{\Sigma_2} \langle \mathbb{B}_2, \mathbb{A}_2\rangle}$ $Z_{II}$

$(\mathbb{B}_1, \mathbb{A}_2; a_I, b_I)$ $\quad (\mathbb{B}_2, \mathbb{A}_3; a_{II}, b_{II})$

(ii) $\tilde{\mathcal{V}}_M = (H(M_I, \Sigma_2) \oplus H(M_{II}, \Sigma_3))[1] \oplus \ldots$

$\downarrow P$

$\mathcal{V}_M = H(M, \Sigma_3)[1] \oplus \ldots$ $\qquad Z_M = P_* \tilde{Z}_M$

$$\xi = (2\pi\hbar)^{\sum_{k=0}^{n} \left(-\frac{1}{4} - \frac{1}{2}k(-1)^k\right)\cdot \dim H^k(M,\partial_{out};E)} \cdot \left(e^{\frac{-\pi i}{2}}\hbar\right)^{\sum_{k=0}^{n}\left(\frac{1}{4}-\frac{1}{2}k(-1)^k\right)\cdot \dim H^k(M,\partial_{out};E)}$$
$$\in \mathbb{C}$$

<u>Poisson sigma model</u>

fix $\pi$ - a Poisson structure on $\mathbb{R}^m$

$M$ - surface (w/ bdry), $n=2$

$\mathcal{H}_\partial, \mathcal{V}_M$ - as before

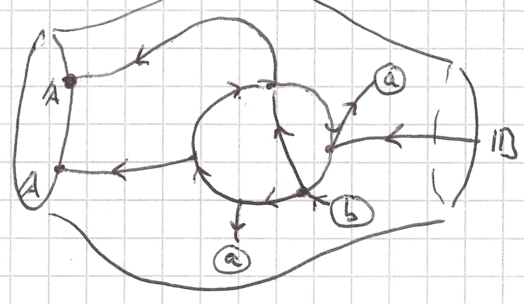
$$S = \int_M \mathbb{B}^i d\mathbb{A}_i + \frac{1}{2}\pi^{ij}(\mathbb{B})\,\mathbb{A}_i \wedge \mathbb{A}_j$$

$$(\mathbb{A},\mathbb{B}) \in \left(\Omega(M)[1] \oplus \Omega(M)\right)\otimes \mathbb{R}^m$$

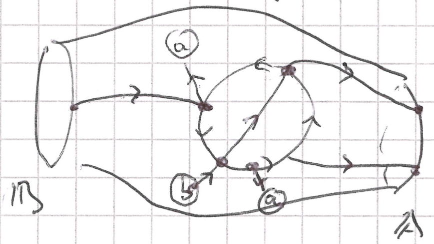
$$Z_M = \xi^{m}\, \tau^m \cdot \exp \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}\,\Gamma|}\, i^{E+V}\, \Phi_\Gamma(\overbrace{\mathbb{A},\mathbb{B}}^{\text{polynomial in}};a,b)$$

$\Gamma$ ← or. conn. graph with $k$ bulk and $(l_{in}, l_{out})$ bdry vertices, univalent

$\Phi_\Gamma$ ← involves $\int_{\mathrm{Conf}_k(M)} \prod_e \eta(\ldots)\cdot \prod_{\text{int vertices}} a(x_v)\cdot b(x_v)$ (univalent bulk)



reflection



$\Omega_{S^1}$ = Standard-ordering quantization of

$$\tilde{S}_{S^1} = \oint_{S^1} \mathbb{B}^i d\mathbb{A}_i + \frac{1}{2}\Pi^{ij}(\mathbb{B})\,\mathbb{A}_i \wedge \mathbb{A}_j$$

$\mathbb{A} \to -i\hbar\frac{\partial}{\partial \mathbb{B}}$ (in $\mathbb{B}$-pol.)

$\mathbb{B} \to -i\hbar\frac{\partial}{\partial \mathbb{A}}$ (in $\mathbb{A}$-pol)

where $\Pi^{ij} = \dfrac{x^i *_\hbar x^j - x^j *_\hbar x^i}{i\hbar} \in C^\infty(\mathbb{R}^m)[[\hbar]]$

Kontsevich's $*$-product induced by $\pi$

<u>Thm</u>: - mQME holds

- changing gauge fixing ($\eta$ & reps of coh.) ⇝ changes $Z$ by $\Delta_\partial\left(\frac{i}{\hbar}\Omega - i\hbar\Delta\right)(\ldots)$
- gluing formula holds ← follows from gluing f-la for propagators