QED 1

1. Free scalar field

1.1 {Classical scalar field (reminder)}

For (M,g) Pseudo-Riemannian, space of fields $F_M = C^\infty(M)$ (fields φ)

action $S: F_M \to \mathbb{R}$, $S(\varphi) = \frac{1}{2}\int_M \left(g^{-1}(d\varphi, d\varphi) + m^2\varphi^2\right) d\,vol$ ← $\sqrt{|\det g|}\, d^n x$

eq. of motion: (Euler-Lagrange eq.) $\underset{\substack{\uparrow \\ \text{metric Laplacian on } C^\infty(M)}}{\Delta}\varphi = -m^2\varphi$

1.2 Specialize to $M = \mathbb{R}^{1,3}$ - Minkowski space

in units $c=1$

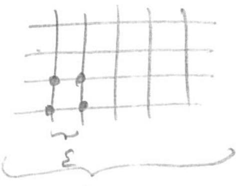
$g = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = \sum g_{ij}\, dx^i dx^j$

$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\quad x^0 = t$

$\Delta_g = +\left(\frac{\partial}{\partial x^0}\right)^2 - \sum_1^3 \left(\frac{\partial}{\partial x^i}\right)^2 = \Box$ - d'Alembertian on $C^\infty(\mathbb{R}^{3,1})$ (wave operator)

$\left(\frac{\partial}{\partial t}\right)^2 - \Delta$

- Replace the space $\Sigma = \mathbb{R}^3$ ($= \mathbb{R}^{0,3}$) with a torus lattice Σ^{discr} with spacing (mesh) ε of perimeter L, $\frac{L}{\varepsilon} = N \in \mathbb{N}$ so Σ^{discr} has N^3 vertices.



L

$$S_{discr}(\varphi) = \int_{\mathbb{R}} L(\varphi, \partial_t\varphi)\, dt$$

$\varphi \in C^\infty(\underset{\substack{\uparrow \\ \text{Set of vertices}}}{\Sigma^{discr}} \times \mathbb{R})$

here the Lagrangian is:

$$\underbrace{L(\varphi, \partial_t\varphi)}_{\text{at a fixed time } t} = \frac{1}{2}\sum_{\text{vertices } v} \varepsilon^3 (\partial_t\varphi_v)^2 - \frac{1}{2}\sum_{\substack{\text{edges} \\ (v_1,v_2)}} \varepsilon^3 \left(\frac{\varphi_{v_1} - \varphi_{v_2}}{\varepsilon}\right)^2 - \frac{1}{2}\sum_{\substack{\text{vertices} \\ v}} \varepsilon^3 m^2 \varphi_v^2$$

$L \in C^\infty(\underset{\varphi}{\mathbb{R}^{\Sigma^{discr}}} \times \underset{\partial_t\varphi}{\mathbb{R}^{\Sigma^{discr}}})$

EL equation: $\left(\frac{\partial^2}{\partial t^2} - \underset{\substack{\uparrow \\ \text{fin. difference Laplacian}}}{\Delta^{discr}} + m^2\right)\varphi = 0$

- Hamiltonian reformulation:

momenta: $\pi_v = \frac{1}{\varepsilon^3}\frac{\partial L}{\partial(\partial_t\varphi_v)} = \partial_t\varphi_v$ $\quad$ Phase space $= T^*\mathbb{R}^{\Sigma^{discr}}$ $\quad$ φ_v - base coord, π_v - fiber coord

$\omega = \sum_v \delta\pi_v \wedge \delta\varphi_v \cdot \varepsilon^3$

Hamiltonian (= Legendre transform of L): $H = \sum_v \varepsilon^3 \pi_v\, \partial_t\varphi_v - L =$

$$= \frac{1}{2}\sum_{\substack{\text{vertices} \\ v}} \varepsilon^3 (\pi_v)^2 + \varepsilon^3 \varphi_v (\Delta^{discr}\varphi)_v + m^2\varepsilon^3\varphi_v^2$$

- collection of harm. oscillators sitting in vertices, interacting over links

Change of coords on the phase space Φ – Fourier transform:

$$\varphi_{\vec{v}} = \sum_{\vec{k} \in \left(\frac{2\pi}{L}\mathbb{Z}_N\right)^{\times 3}} \frac{1}{L^3} e^{i(\vec{k}\cdot\vec{x})} \tilde{\varphi}_{\vec{k}}, \qquad \vec{v} = (x^1,x^2,x^3) \in \left(\varepsilon\cdot\left(\mathbb{Z}/_{N\mathbb{Z}}\right)\right)^{\times 3}$$

$\underbrace{\left(\frac{2\pi}{L}\mathbb{Z}_N\right)^{\times 3}}_{W}$ "wave vectors"

$$\pi_{\vec{v}} = \sum_{\vec{k}} \frac{1}{L^3} e^{i(\vec{k}\cdot\vec{x})} \tilde{\pi}_{\vec{k}} \qquad \omega = \sum_{\vec{k}} \frac{1}{L^3} \delta\tilde{\pi}_{-\vec{k}} \wedge \delta\tilde{\varphi}_{\vec{k}}$$

Reality of φ_v, π_v $\Rightarrow \varphi_{-\vec{k}} = \overline{\varphi_{\vec{k}}}$, $\pi_{-\vec{k}} = \overline{\pi_{\vec{k}}}$

$$H = \frac{1}{2}\sum_{\vec{k}} \frac{1}{L^3}\Big(\pi_{-\vec{k}}\pi_{\vec{k}} + \overbrace{k^2 \varphi_{-\vec{k}}\varphi_{\vec{k}} + m^2 \varphi_{-\vec{k}}\varphi_{\vec{k}}}^{(k^2+m^2)\cdot\varphi_{-\vec{k}}\varphi_{\vec{k}}}\Big)$$

- field decoupled into pairs of non-interacting harmonic oscillators corresp. to $(\vec{k},-\vec{k})$ and one special $\vec{k}=0$.

frequencies: $\omega_k = \sqrt{k^2+m^2}$

1.3. Canonical quantization. $\varphi_{\vec{k}} \longmapsto \hat{\varphi}_{\vec{k}}$, $\pi_{\vec{k}} \longmapsto \hat{\pi}_{-\vec{k}}$ - operators on some $\mathcal{H}$

Heisenberg's canonical commutation relations: $[\hat{\varphi}_k, \hat{\pi}_\ell] = i\hbar L^3 \delta_{k+\ell}\cdot\mathbb{1}$ $(*)$

Rem with a change of coord on Φ $\varphi_{\vec{k}} = \dfrac{\varphi^{Re}_{\vec{k}} + i\varphi^{Im}_{\vec{k}}}{\sqrt{2}}$, $\pi_{\vec{k}} = \dfrac{\pi^{Re}_{\vec{k}} + i\pi^{Im}_{\vec{k}}}{\sqrt{2}}$ reality: $\varphi^{Re}_{-\vec{k}} = \varphi^{Re}_{\vec{k}}$, $\varphi^{Im}_{-\vec{k}} = \varphi^{Im}_{k}$ similarly for π

$\varphi_{\vec{k}}\varphi_{\vec{k}} = \frac{1}{2}\left((\varphi^{Re}_k)^2 + (\varphi^{Im}_k)^2\right)$, $\pi_{-\vec{k}}\pi_{\vec{k}} = \frac{1}{2}\left((\pi^{Re}_{\vec{k}})^2 + (\pi^{Im}_{\vec{k}})^2\right)$

$$\omega = \sum_{\vec{k}/_{\vec{k}\sim-\vec{k}}} \frac{1}{L^3}\left(\delta\pi^{Re}_k \wedge \delta\varphi^{Re}_k + \delta\pi^{Im}_k \wedge \delta\varphi^{Im}_k\right) \Rightarrow \text{CCR: } [\hat{\varphi}^{Re}_k, \hat{\pi}^{Re}_\ell] = i\hbar L^3 \delta_{k\ell}\cdot\mathbb{1}$$

$k,\ell \in W/_{\mathbb{Z}_2}$ likewise for Im.

creation/annihilation operators: $\hat{a}^{\pm}_k = \dfrac{\sqrt{\omega_k}\hat{\varphi}_k \mp i\,\hat{\pi}_k/\sqrt{\omega_k}}{\sqrt{2}}$ $\Leftrightarrow$ $\hat{\varphi}_k = \dfrac{\hat{a}_k + \hat{a}^+_k}{\sqrt{2\omega_k}}$, $\hat{\pi}_k = \sqrt{\omega_k}\dfrac{\hat{a}_k - \hat{a}^+_k}{i\sqrt{2}}$

reality: $(\hat{a}_k)^{+} = \hat{a}^{+}_{k}$ hermitian conjugation

notation: $a^- = a$

CCR $(*)$: $[\hat{a}_k, \hat{a}^+_\ell] = L^3\hbar\cdot\delta_{k+\ell}\mathbb{1}$

$([\hat{a}_k, \hat{a}_\ell] = [\hat{a}^+_k, \hat{a}^+_\ell] = 0)$

Quantum Hamiltonian: $\hat{H} = \frac{1}{2}\sum_{\vec{k}} \frac{1}{L^3}\Big(\hat{\pi}_{-\vec{k}}\hat{\pi}_{\vec{k}} + \overbrace{(k^2+m^2)}^{\omega_k^2}\hat{\varphi}_{-\vec{k}}\hat{\varphi}_{\vec{k}}\Big)$

$$= \sum_{\vec{k}} \frac{1}{L^3}\,\omega_k\cdot\left(\hat{a}^+_{-\vec{k}}\hat{a}_{\vec{k}} + \frac{\hbar}{2}\hat{\mathbb{1}}\right)$$

$[\hat{H}, \hat{a}^{\pm}_k] = \pm\hbar\omega_k \hat{a}^{\pm}_k$, thus $\hat{a}^{\pm}_k$ are the "ladder operators"

Constructing the space of states (Fock representation) ($\sim$ Verma module for the algebra $\mathrm{Heis} = \langle \{\hat{a}^{\pm}_k\}, \hat{\mathbb{1}}\rangle/_{CCR}$)

$|0\rangle \in \mathcal{H}$ - the "vacuum", satisfying $a_k|0\rangle = 0$ $\forall k$ - highest vector (lowest)

$\hat{H}|0\rangle = C\hat{\mathbb{1}} = \sum \frac{1}{L^3}\frac{\hbar}{2}\cdot\hat{\mathbb{1}}$

basis in $\mathcal{H}$: $|\vec{k}_1,\dots,\vec{k}_n\rangle = a^+_{\vec{k}_1}\cdots a^+_{\vec{k}_n}|0\rangle$ - some k-s can repeat, $n\in\mathbb{N}_0$

Action of Heis defined uniquely by CCR

e.g. $a^+_{\vec{k}}|\vec{\ell}\rangle = |\vec{k},\vec{\ell}\rangle$

$a^-_{\vec{k}}|\vec{\ell}\rangle = a^-_k a^+_\ell|0\rangle = (\hbar L^3\delta_{k+\ell} - a^+_\ell a^-_k)|0\rangle$

$= \hbar L^3\delta_{k+\ell}|0\rangle$

There exists a unique sesquilinear form $(\,,\,)$ on $\mathcal{H}$ such that (1) $\langle 0|0\rangle = 1$ (i.e. $(|0\rangle, |0\rangle)$); (2) the operators $a_{\vec k}$ and a^+_{-k} are adjoint to each other.

Exercise: calculate the norm of state $|k_1 \dots k_n\rangle$.

- Relation $[\hat H, \hat a^+_k] = \hbar\omega_k \hat a^+_k$ implies that

$$\hat H\, |\vec k_1 \dots \vec k_n\rangle = \Big(\sum_{j=1}^{n} \hbar\omega_{k_j} + C\Big)|\vec k_1 \dots \vec k_n\rangle$$

interpretation: $|\vec k_1 \dots \vec k_n\rangle$ is a state with n-particles having momenta $\hbar\vec k_1 \dots \hbar\vec k_n$ and energy $\hbar\omega_{k_1} \dots \hbar\omega_{k_n}$ resp.; C is the vacuum energy.

$\hat H$ is the total energy operator

total momentum of the field

classically: $P_i = \int_{\Sigma = \mathbb{R}^3} \pi\, \partial_i \varphi \, d\mathrm{vol}_\Sigma$ $\xrightarrow[\text{quantization}]{\text{can.}}$ $[\hat P_i, \hat a^\pm_k] = \hbar k_i\, \hat a^\pm_k$ (i-th component)

- Nötler current assoc. with spacial translations.

$\Rightarrow |\vec k_1 \dots \vec k_n\rangle$ is an eigenstate of $\hat P_i$ with eigenvalue $\hbar(\vec k_1 + \dots + \vec k_n)_i$

Thus (application of) $a^+_{\vec k}$ "creates" a particle ("quantum of the field") with energy-momentum 4-vector $p = (\hbar\omega_k, \hbar\vec k)$

note: $p^2 = (p,p) = \hbar^2(\omega_k^2 - k^2) = \hbar^2 m^2$ – corresponds to a particle with "Resting frame mass" equal to $\hbar m$.

Rem We also can do Schrödinger quantization, without going to creation/annihil. operators

$\mathcal{H}^{\text{Schröd}} = L^2(\underbrace{\mathbb{R}^{\Sigma^{\text{discr}}}}_{\text{config space}})$ $\ni \psi(\{\varphi_v\})$

= base of phase space

with field operators $\hat\varphi_{\vec x} : \psi \longmapsto \varphi_{\vec x}\cdot\psi$ – multiplication op.

and $\hat\pi_{\vec x} : \psi \longmapsto -\frac{i\hbar}{\varepsilon^3}\frac{\partial}{\partial\varphi_{\vec x}}\psi$ – derivation.

Exercise: construct an explicit isomorphism $\mathcal{H}^{\text{Schröd}} \xleftarrow{\sim} \mathcal{H}^{\text{Fock}}$ of Hilbert spaces (uses Hermite polynomials)

Limit $\varepsilon \to 0$, $L \to \infty$ (not necessary): $\mathcal{H}^{\text{Fock}} = \bigoplus_{n \geq 0} \mathcal{H}^{\text{Fock}}_n$, $\mathcal{H}^{\text{Fock}}_0 = \mathbb{C}\cdot|0\rangle$

$$\mathcal{H}^{\text{Fock}}_n \ni \int_{(\mathbb{R}^3)^n} d^3\vec k_1 \dots d^3\vec k_n \cdot \underbrace{\psi_n(\vec k_1 \dots \vec k_n)}_{\text{symmetric in } k_1 \dots k_n} \cdot \underbrace{|\vec k_1 \dots \vec k_n\rangle}_{\hat a^+_{k_1} \dots \hat a^+_{k_n}|0\rangle}$$

– n-particle state

W

$|\psi_n(\vec k_1 \dots \vec k_n)|^2$ = probability density to find

"cobordism"
$\Sigma \times [t_{in}, t_{out}]$

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Evolution operator $\hat{U}(t) = \exp\left(-\frac{i}{\hbar} t \hat{H}\right)$

is diagonal in Fock representation:

$$\hat{U}(t)\,|\vec{k}_1 \dots \vec{k}_n\rangle = \prod_{j=1}^{n} e^{-i\omega_{k_j} t} \cdot |\vec{k}_1 \dots \vec{k}_n\rangle$$

↑ phase attained by j-th particle over time t.

$\longleftrightarrow \hat{U}(t): \mathcal{H} \to \mathcal{H}$

"$\hat{H}$"

In QM:

Schrödinger picture	Heisenberg picture
states ψ depend on time, $\psi(t_{out}) = \hat{U}(t_{out} - t_{in}) \cdot \psi(t_{in})$	state $\psi \in \mathcal{H}$ is time-indep
observables are fixed operators $\hat{O}$ on $\mathcal{H}$ (time-indep)	O depends on t via $O_{t_{out}} = U^{-1} O_{t_{in}} U$, $U = \hat{U}(t_{out} - t_{in})$ $\longrightarrow \frac{d}{dt}\hat{O}_t = \frac{i}{\hbar}[\hat{H}, \hat{O}_t]$

compatibility:

$t_{in} = 0$, $t_{out} = t$

$$\langle O_t \rangle_\psi = \langle \psi_t^{Schr} | O | \psi_t^{Schr} \rangle = \langle \psi | O_t^{Heis} | \psi \rangle$$

Commutator of (Heisenberg) field operators ($\neq$ "propagator")

$$[\hat{\varphi}(x), \hat{\varphi}(y)] = \int \frac{d^3\vec{k}}{(2\pi)^3} \cdot \frac{d^3\vec{\ell}}{(2\pi)^3} \underbrace{\left[\frac{\hat{a}_k(t) + \hat{a}_k^+(t)}{\sqrt{2\omega_k}}, \frac{\hat{a}_\ell(s) + \hat{a}_\ell^+(s)}{\sqrt{2\omega_\ell}}\right]}_{\hbar \frac{1}{2\sqrt{\omega_k \omega_\ell}}\left(e^{-i\omega_k t + i\omega_\ell s}\delta(\vec{k}+\vec{\ell}) - e^{i\omega_k t - i\omega_\ell s}\delta(\vec{k}+\vec{\ell})\right)\cdot \mathbb{1}} e^{i(\vec{k}\cdot\vec{x} + \vec{\ell}\cdot\vec{y})} \overset{(=)}{}$$

$x = (t, \vec{x})$, $y = (s, \vec{y})$ — from CCR for a, a^+

$a_k^+(t) = U_t^{-1} a_k^+ U_t = e^{i\omega_k t} a_k^+$

$a_k^-(t) = U_t^{-1} a_k^- U_t = e^{-i\omega_k t} a_k^-$

$$\overset{(=)}{} \hbar \cdot \mathbb{1} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega_k(s-t) + i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{i\omega_k(t-s) + i\vec{k}\cdot(\vec{x}-\vec{y})}}{2\omega_k} = \mathbb{1} \cdot (D(x,y) - D(y,x))$$

4-vector $(\omega_k, \vec{k})$

$$D(x,y) := \langle 0 | \hat{\varphi}(x) \hat{\varphi}(y) | 0 \rangle = \hbar \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{e^{-i(k^{(4)}, x - y)_{Mink}}}{2\omega_k}$$

- $D(x,y)$ is Lorentz-invariant!

$\Rightarrow$ (1) for $\|x - y\|^2 > 0$ (timelike separation), $\|x - y\|^2 =: \tau^2$

$$D(x,y) = \hbar \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{e^{-i\omega_k \tau}}{2\omega_k} = \hbar \cdot \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{x^2\,dx}{2\sqrt{x^2 + m^2}} e^{-i\sqrt{x^2+m^2}\,\tau} \underset{\omega = \sqrt{x^2+m^2}}{=} \frac{\hbar}{(2\pi)^2} \int_m^\infty d\omega \sqrt{\omega^2 - m^2}\, e^{-i\omega\tau}$$

$$= \dots \underset{\tau \to \infty}{\sim} e^{-im\tau}$$

(2) for $\|x - y\|^2 < 0$, $\|x-y\|^2 = -r^2$, $D(x,y) \underset{r \to \infty}{\sim} e^{-mr}$ (is <u>real</u> and exp. decaying)

(3) for $\|x - y\|^2 < 0$, $D(x,y) = D(y,x)$ — from existence of a Lorentz transform interchanging x and y.

Exercise: calculate $D(x,y)$ explicitly.

We have $[\hat{\varphi}(x), \hat{\varphi}(y)] = 0$ for $\|x-y\|^2 < 0$ — spacelike separated

thus measurements of the field at points are <u>independent</u>.

$\Rightarrow$ agrees with causality.

<u>Rem</u>: $D(x,y)$ is singular at $x \to y$ (in the limit $\varepsilon \to 0$)

this is an effect of $\hat{\varphi}(x)$ becoming an operator-valued <u>distribution</u> on Σ, (or $\Sigma \times \mathbb{R}$ in Heis. picture) supported at x

$\rightsquigarrow$ product of distributions becomes <u>singular</u> when supports overlap – in this case, $x = y$.

• ~~$D(x,y)$ is the Green's function for operator~~ $\square + m^2$

(recall: $S(\varphi) = -\frac{1}{2} \int_{\mathbb{R}^{1,3}} \varphi \cdot \square \varphi \cdot d\mathrm{vol}$

• ~~$D(x,y) = \hbar \int d^4k \, e^{i(x-y,k)_{(4)}} \tilde{D}(k)$~~ $D(y,x)$

where $\tilde{D}(k) = \frac{1}{k^2 - m^2 + i0}$

$$D_R(x,y) \equiv \vartheta(x^0 - y^0) \langle 0 | [\hat{\varphi}(x), \hat{\varphi}(y)] | 0 \rangle \overset{?}{=} \vartheta(x^0-y^0)\, \hbar \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right) =$$

$$= \hbar \int \frac{d^4k}{(2\pi)^4} \frac{i\hbar}{k^2 - m^2 + i0} e^{-ik\cdot(x-y)}$$

displaced poles from real line in k^0 $\rightsquigarrow$ require to choose the contour for integrating over k^0.

for $x^0 > y^0$ close below

for $x^0 < y^0$ – close above and get 0.

$D_R(x,y)$ – <u>retarded</u> Green's function for the operator $\square + m^2$:

$$(\square + m^2) D_R(x,y) = -i\hbar \delta^{(4)}(x-y)$$

Feynman propagator: choose the contour

for the int. over $k^{(0)}$:

$$D_F = \hbar \int \frac{d^4k}{(2\pi)^4} \frac{i\hbar}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)}$$

$$= \vartheta(x^0 - y^0) \langle 0 | \hat{\varphi}(x) \hat{\varphi}(y) | 0 \rangle + \vartheta(y^0 - x^0) \langle 0 | \hat{\varphi}(y) \hat{\varphi}(x) | 0 \rangle$$

$$=: \langle 0 | T \varphi(x) T(y) | 0 \rangle$$

"time-ordering"

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2. Interacting theory

Consider the scalar theory with "φ^4" interaction:

$$S(\varphi)=\int_{M=\mathbb{R}^{1,3}} d\mathrm{vol}\left(\frac{1}{2}\sum_{\mu,\nu=0}^{3}\eta^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi-\frac{m^2}{2}\varphi^2-\frac{\lambda}{\underbrace{24}_{=4!}}\varphi^4\right)=S_0(\varphi)+S_{int}(\varphi)$$

Hamiltonian $\underset{\substack{\in\\ \mathrm{Fun}(\Phi)\\ \| \\ C^\infty(\Sigma)\oplus C^\infty(\Sigma)}}{H(\varphi,\pi)} = \underbrace{H_0(\varphi,\pi)}_{\int_\Sigma d^3x\left(\frac{\pi^2}{2}+\sum_{i=1}^{3}\frac{1}{2}(\partial_i\varphi)^2-\frac{m^2}{2}\varphi^2\right)} + \underbrace{H_{int}(\varphi)}_{\frac{\lambda}{4!}\int_\Sigma d^3x\,\varphi^4}$

$\Sigma=\mathbb{R}^3$

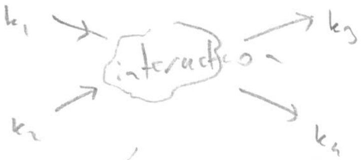
Quantization: Use same H as in free theory.

$$\hat H=\hat H_0+\frac{\lambda}{4!}\int_\Sigma d^3x\,\underbrace{:\hat\varphi(x)^4:}_{\text{ordering prescription}}=\hat H_0+\hat H_{int}$$

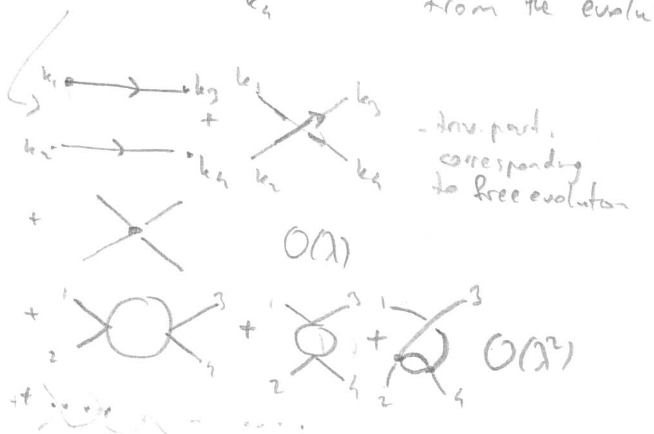
normal ordering: a word w in operators $\{a_k,a_k^+\}$ $\longmapsto$ word $:w:$ obtained by putting all a^+ to the left and all a^- to the right.

this is a map $\mathbb{C}\langle\{\hat a_k\},\{\hat a_k^+\}\rangle\longrightarrow\mathbb{C}\langle\{\hat a_k\},\{\hat a_k^+\}\rangle$
before quotienting by relations.

We are interested in the evolution operator $U(t_{out}-t_{in})=e^{-\frac{i}{\hbar}\hat H(t_{out}-t_{in})}:\mathcal{H}\to\mathcal{H}$
in the asymptotics $t_{in}\to-\infty$, $t_{out}\to+\infty$ as a formal series in λ.
E.g. the block $U^{2\to2}:\mathcal{H}_2\to\mathcal{H}_2$ corresponds to 2-particle scattering



we want to "substract" the trivial part, corresp. to $\hat H_0$ from the evolution



"Interaction picture"

$\tilde U(t,t_0):=e^{\frac{i}{\hbar}\hat H_0(t-t_0)}e^{-\frac{i}{\hbar}\hat H(t-t_0)}$

$\varphi(t,\vec x)=\tilde U^{-1}(t,t_0)\,\underbrace{\varphi_I(t,\vec x)}_{U_0^{-1}\varphi(t_0,\vec x)U_0}\,\tilde U(t,t_0)$

$\tilde U$ satisfies

$i\hbar\frac{\partial}{\partial t}\tilde U(t,t_0)=\hat H_I(t)\,\tilde U(t,t_0)$ $\quad(*)$

where $\hat H_I(t):=U_0^{-1}H_{int}U_0=\int d^3x\,\frac{\lambda}{4!}:\hat\varphi_I^4:$

Solution of $(*)$: $\tilde U(t,t_0)=\mathrm{Pexp}\left(-\frac{i}{\hbar}\int_{t_0}^{t}\hat H_I(t')dt'\right)=\sum_{n\geq0}\left(-\frac{i}{\hbar}\right)^n\int_{t_0<t_1<\cdots<t_n<t}dt_1\cdots dt_n\,\hat H_I(t_n)\cdots\hat H_I(t_1)=$

$=\sum_{n\geq0}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\int_{[t_0,t]^n}dt_1\cdots dt_n\,\underset{\substack{\uparrow\\ \text{time-ordering}}}{T}\hat H_I(t_n)\cdots\hat H_I(t_1)$

Comparing T-ordering to $::$-ordering

$$T\hat\varphi_I(x)\hat\varphi_I(y)=:\hat\varphi_I(x)\hat\varphi_I(y):+\underset{\substack{\uparrow\\ \text{Feynman propagator}}}{D_F}(x,y)\hat{\mathbb{1}}$$

notation: $\underbrace{\varphi_I(x)\varphi_I(y)}:=D_F(x,y)\hat{\mathbb{1}}$

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Wick's lemma

$$T\,\hat{\varphi}_I(x_1)\dots\hat{\varphi}_J(x_n) = :\hat{\varphi}_J(x_1)\dots\hat{\varphi}_I(x_N) + \text{all possible contractions}:$$

E.g. $T\varphi_1\varphi_2\varphi_3\varphi_4 = :\varphi_1\varphi_2\varphi_3\varphi_4:$

$$+ :\underbrace{\varphi_1\varphi_2}\varphi_3\varphi_4: + :\varphi_1\underbrace{\varphi_2\varphi_3}\varphi_4: + :\varphi_1\varphi_2\underbrace{\varphi_3\varphi_4}: + :\underbrace{\varphi_1\varphi_2\varphi_3}\varphi_4: + :\varphi_1\underbrace{\varphi_2\varphi_3\varphi_4}: + :\underbrace{\varphi_1\varphi_2\varphi_3\varphi_4}:$$

$$+ :\underbrace{\varphi_1\varphi_2}\underbrace{\varphi_3\varphi_4}: + :\varphi_1\varphi_2\varphi_3\varphi_4: + :\varphi_1\varphi_2\varphi_3\varphi_4:$$

In particular: $\langle 0|T\varphi_1\varphi_2\varphi_3\varphi_4|0\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)$.

Feynman graphs