Atiyah's definition of n-dimensional TQFT ('88)

DATA

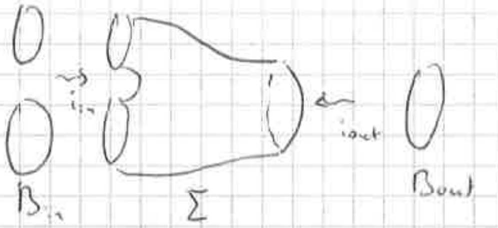
(1) closed (oriented) $(n-1)$-manifolds $\longrightarrow$ vector spaces over $\mathbb{C}$

$B \longmapsto \mathcal{H}_B$ ("space of states")

(2) (oriented) n-cobordisms Σ $\longrightarrow$ linear maps between spaces of states

$(\Sigma, B_{in}, B_{out}, i_{in}, i_{out}) \longmapsto Z_\Sigma : \mathcal{H}_{B_{in}} \to \mathcal{H}_{B_{out}}$

$\partial \Sigma = i_{in}(\overline{B_{in}}) \amalg i_{out}(B_{out})$



(3) a (projective) representation ~~$\rho_B : \mathrm{Diff}_+(B) \to \mathrm{End}(\mathcal{H}_B)$~~

$\rho : (\text{diffeo } B \xrightarrow{f} B') \longmapsto \text{iso } \rho(f) : \mathcal{H}_B \to \mathcal{H}_{B'}$

(i) multiplicativity: $\cdot\ \mathcal{H}_{B_1 \amalg B_2} = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$

$\cdot\ Z_{\Sigma_1 \amalg \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2} : \mathcal{H}_{\Sigma_1^{in}} \otimes \mathcal{H}_{\Sigma_2^{in}} \to \mathcal{H}_{\Sigma_1^{out}} \otimes \mathcal{H}_{\Sigma_2^{out}}$

$\cdot\ \mathcal{H}_\emptyset = \mathbb{C}$

(ii) functoriality

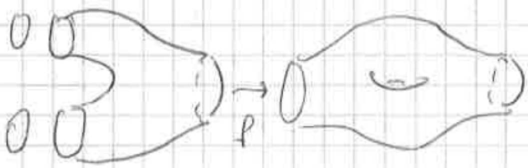
$\Sigma_1 : B_1^{in} \to B_1^{out}$

$\Sigma_2 : B_2^{in} \xrightarrow{} B_2^{out}$, $\quad f : B_1^{out} \xrightarrow{\sim} B_2^{in}$

$\xrightarrow{\text{gluing}}$ $\Sigma = \Sigma_1 \cup_f \Sigma_2$

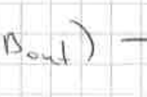
axiom:

$$\begin{tikzcd} \mathcal{H}_{B_1^{in}} \arrow[r, "Z_\Sigma"] \arrow[d, "Z_{\Sigma_1}"'] \arrow[dr, phantom, "\square"] & \mathcal{H}_{B_2^{out}} \\ \mathcal{H}_{B_1^{out}} \arrow[r, "\rho(f)"] & \mathcal{H}_{B_2^{in}} \arrow[u] \end{tikzcd}$$



(iii) normalization

$Z_{B \times [0,1]} = \mathrm{id} : \mathcal{H}_B \to \mathcal{H}_B$ [specifics of a topological theory]



(iv) for a orientation preserving diffeo $f : (\Sigma, B_{in}, B_{out}) \to (\Sigma', B'_{in}, B'_{out})$,

$$\begin{tikzcd} \mathcal{H}_{B_{in}} \arrow[r, "Z_\Sigma"] \arrow[d, "\rho(f|_{B_{in}})"'] \arrow[dr, phantom, "\square"] & \mathcal{H}_{B_{out}} \arrow[d, "\rho(f|_{B_{out}})"] \\ \mathcal{H}_{B'_{in}} \arrow[r, "Z_{\Sigma'}"'] & \mathcal{H}_{B'_{out}} \end{tikzcd}$$

Rem: in particular, $\mathrm{Diff}(\Sigma, \partial\Sigma)$ act on Z trivially.

(v) symmetry:

$$\begin{tikzcd} B \amalg B' \arrow[d, "\wr", "\text{diffeo}"'] & & \mathcal{H}_B \otimes \mathcal{H}_{B'} \arrow[d, "\wr", "\text{iso}"'] \\ B' \amalg B \arrow[r, phantom, "\overset{\rho}{\longrightarrow}"] & & \mathcal{H}_{B'} \otimes \mathcal{H}_B \end{tikzcd}$$

Rem 1) In the categorical language: an n-TQFT is a functor of symmetric monoidal categories

$(\mathrm{Cob}_n, \underset{\cup}{\circ}, \underset{B\times[0,1]}{id_B}, \underset{\amalg}{\otimes}, \underset{\emptyset \in \mathrm{Ob}}{\mathbb{1}}) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$

- Ob = closed $(n-1)$-mfds
- Hom = cobordisms

- Ob = vector spaces
- Hom = linear maps

2) can consider values in k-mod (instead of $\mathrm{Vect}_{\mathbb{C}}$) for a comm. ring k

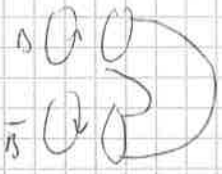
3) can allow more structure on cobordisms (e.g. framing, spin-structure etc.)

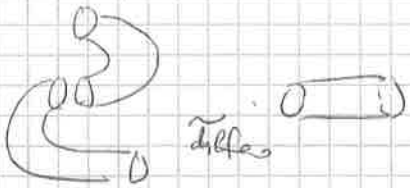
4) n-mfld Σ with boundary $\partial\Sigma$ can be considered as a cob. $\emptyset \xrightarrow{\Sigma} \partial\Sigma$; $Z(\Sigma) \in \mathcal{H}_{\partial\Sigma}$ - "vacuum vector"

Consequences of axioms

- For Σ n-dim. closed, $\Sigma : \emptyset \to \emptyset$ induces $Z_\Sigma : \mathbb{C} \to \mathbb{C}$ - multiplication by a complex number - a diff.-invariant of Σ.

- $B \amalg \bar{B}$ cobordant to $\emptyset$ $\Rightarrow \langle , \rangle : \mathcal{H}(B) \otimes \mathcal{H}(\bar{B}) \to \mathbb{C}$ - $\underbrace{\text{non-degenerate}}$ pairing





- If $\Sigma : B \to B'$ and $\varphi : B' \to B$ - diffeo, and $\tilde{\Sigma} = \Sigma / \underset{\overset{\cap}{B'}}{x} \sim \underset{\overset{\cap}{B}}{\varphi(x)}$ - closed

then $Z(\tilde{\Sigma}) = tr\, \rho(\varphi) Z_\Sigma$

- in particular, for a ~~mapp cylinder~~ "torus" $S^1 \times B$, $Z_{S^1\times B} = \dim \mathcal{H}_B \in \mathbb{N}$

and for a "mapping torus" $[0,1]\times B / (1,x) \sim (0, \varphi(x))$, $Z = tr\, \rho(\varphi)$

- $\mathcal{H}_B$ are finite-dimensional

5) If ρ extends ~~from $\mathrm{Diff}_+(B)$ to $\mathrm{Diff}(B)$~~ to non-or.-preserving diffeo ("TQFT invariant wrt change of orientation")

then there is $\sigma_B = \underset{(\mathbb{C}\text{-antilinear})}{\rho(\bar{B} \xrightarrow{id} B)} : \mathcal{H}_{\bar{B}} \to \mathcal{H}_B$

composing σ_B with $\mathcal{H}_B = \mathcal{H}_{\bar{B}}^*$, we get a Hermitian form on $\mathcal{H}_B$

6) - state on B: $s \in \mathrm{End}(\mathcal{H}_B)$
  - Hermitian
  - positive (eigenvalues ≥ 0)
  - $tr(s) = 1$

- pure states
  - one-dimensional projectors $s = P_\psi = \psi^* \otimes \psi^*$

$\mathcal{H}_B / S^1 \simeq \{\text{pure states}\}$

TQFT in low dimensions

n = 1 · classified by $\dim \mathcal{H}_{pt}$

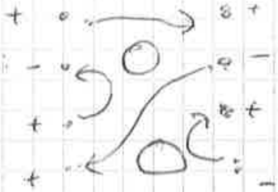
$\mathcal{H}_{pt^+} = V \qquad \mathcal{H}_{pt^-} = V^* \qquad Z_{\bullet_+ \to \bullet_+} = id : V \to V$

$\rightsquigarrow \mathbb{C} \xrightarrow[\text{coev}]{} V \otimes V^*$

$\rightsquigarrow V^* \otimes V \xrightarrow[\text{ev}]{} \mathbb{C}$

$\emptyset \bigcirc \emptyset \rightsquigarrow \mathbb{C} \xrightarrow[\cdot \dim V]{} \mathbb{C}$

generic 1-Cob:



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n = 2 classified by Frobenius algebras $(A, \mu, 1, \varepsilon, \Delta, \langle , \rangle)$

counit $A \to \mathbb{C}$

sym non-degenerate pairing $\langle a, b \rangle = \varepsilon(ab)$

$\mathcal{H}_{S^1} = A$

$\mu = Z\Big($  $\Big) : A \otimes A \to A$

$\langle , \rangle = Z\Big($  $\Big) : A \otimes A \to \mathbb{C}$

$1 = Z(\text{(}\!\!\bigcirc\!\!\text{)}), \; \varepsilon = Z(\text{(}\!\!\bigcirc\!\!\text{)})$

$\mu^* : A^* \to A^* \otimes A^*$
$\quad\;\; \wr\!\wr \qquad \wr\!\wr \quad \wr\!\wr$
$\Delta : A \to A \otimes A$

$Z\Big($  $\Big) = \varepsilon(\mu \circ \Delta)^g 1 \in \mathbb{C}$

genus g

Exercise 1) check that, G - fin group,

$B \longmapsto \mathrm{Span}_{\mathbb{Q}}\left(\mathrm{Hom}(\pi_1(B), G)/_G\right)$

$\Sigma \longmapsto Z_\Sigma(\alpha_1, \alpha_2) = \dfrac{\#\{\gamma \in \mathrm{Hom}(\pi_1(\Sigma), G) \mid [\gamma|_{B_1}] = \alpha_1, [\gamma|_{B_2}] = \alpha_2\}}{|G|}$

is a TQFT

2) show that for n=2, corresponding Frobenius algebra is $\underset{\text{center}}{Z}(\underbrace{\mathbb{Q}[G]}_{\text{group ring}})$

TQFT 1/4

draft

• ~~Example of a TQFT; vertex model of stat. mechanics.~~

In statistical mechanics: finite set of states ~~$\mathcal{S}$~~, probability measure on states

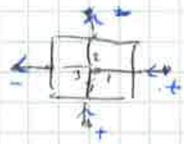
$$\mathrm{Prob}(\sigma_0) = \frac{e^{-\frac{E(\sigma_0)}{kT}}}{\sum_{\text{states } \sigma} e^{-\frac{E(\sigma)}{kT}}}$$

$\sum_{\text{states } \sigma} e^{-\frac{E(\sigma)}{kT}} \leftarrow Z(T)$ - the partition function

vertex model on graphs

data: X - finite set,
$w_n : \underbrace{X \times \dots \times X}_{n} \to \mathbb{C}$ ↙ assume $w_1 = 1$

Γ - ~~oriented~~ oriented graph with ordering of adjacent edges at each vertex

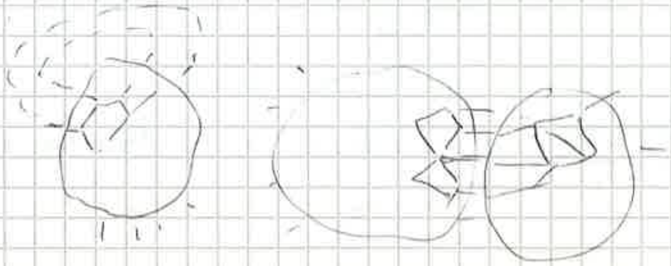


- "$\partial\Gamma$" = set of 1-valent vertices,
- $E(\partial\Gamma)$ - set of ~~vert~~ edges adjacent to $\partial\Gamma$
- $\mathcal{H}(\partial\Gamma) = \mathrm{Span}_{\mathbb{C}}\{\varphi : E(\partial\Gamma) \to X\}$, $\dim \mathcal{H}(\partial\Gamma) = |X|^{|E(\partial\Gamma)|}$

vector $Z(\Gamma) \in \mathcal{H}(\partial\Gamma)$

$$Z(\Gamma)_{\{a(e)\}_{e\in\partial\Gamma}} = \sum_{\substack{\varphi : E(\Gamma) \to X \\ \varphi|_{\partial\Gamma} = a}} \prod_{v \in V(\Gamma)} \underbrace{w(a(e_1), \dots, a(e_n))}_{e_1 \dots e_n \text{ - edges adjacent to } v}$$

$$Z(\Gamma) = \sum_{\{a_i\}} Z(\Gamma)_{\{a_i\}} \underbrace{[\{a_i\}]}_{\text{basis vector in } \mathcal{H}(\partial\Gamma)}$$



~~$\partial\Gamma = \partial'\Gamma \sqcup \partial_0\Gamma \sqcup \partial$~~

$\partial\Gamma$ - oriented 0-dim space $\begin{smallmatrix} + & + & - & + & - \\ \downarrow & \downarrow & \uparrow & \downarrow & \uparrow \end{smallmatrix}$

• Pairing $\langle\, ,\rangle : \mathcal{H}(\overline{\partial\Gamma}) \otimes \mathcal{H}(\partial\Gamma) \to \mathbb{C}$

$$\langle \{a_i\}_{i\in\overline{\partial\Gamma}}, \{b_i\}_{i\in\partial\Gamma} \rangle = \prod_{i\in\partial\Gamma} \delta_{a_i, b_i}$$

- $\mathcal{H}(\emptyset) = \mathbb{C}$
  $\mathcal{H}(N_1 \sqcup N_2) = \mathcal{H}(N_1) \otimes \mathcal{H}(N_2)$

• $Z(\Gamma_1 \sqcup \Gamma_2) = Z(\Gamma_1) \otimes Z(\Gamma_2) \in \mathcal{H}(\partial\Gamma_1) \otimes \mathcal{H}(\partial\Gamma_2)$

- If $\partial\Gamma = \partial'\Gamma \sqcup \partial_0\Gamma \sqcup \widetilde{\partial_0\Gamma}$, or.-preserving bijection $\varphi : \widetilde{\partial_0\Gamma} \xrightarrow{\sim} \overline{\partial_0\Gamma}$

Γ_φ = result of gluing $\partial_0\Gamma$ and $\widetilde{\partial_0\Gamma}$ along φ ($\partial'\Gamma$)

$$\langle\ \rangle_\varphi : \mathcal{H}(\partial\Gamma) = \mathcal{H}(\partial'\Gamma) \otimes \mathcal{H}(\partial_0\Gamma) \otimes \mathcal{H}(\widetilde{\partial_0\Gamma})$$
$$\downarrow \mathrm{id} \otimes \langle\, ,\rangle$$
$$\mathcal{H}(\partial'\Gamma)$$

then $\langle Z(\Gamma)\rangle_\varphi = Z(\Gamma_\varphi) \in \mathcal{H}(\partial'\Gamma)$

for $w(a_1, \dots, a_n) = e^{-\frac{E(a_1, \dots, a_n)}{kT}}$, this is a model of stat. mechanics

... probabilities of states $\{b\}$ will a fixed ...

Dijkgraaf-Witten theory

Ref: "Topological gauge theory and group cohomology", Comm. in Math. Phys. 129 (1990) 393-429

fix G - finite group

principal G bundles $\begin{matrix} E \circlearrowleft G \\ \downarrow \\ \Sigma \end{matrix}$ have a unique connection, which is automatically flat

$$\{G\text{-bundles over } \Sigma\}/\text{isomorphisms} \xrightarrow{\text{monodromy}} \gamma : \pi_1(\Sigma, *) \to G \text{ - homomorphisms}/\gamma \sim h\gamma h^{-1}$$

$h \in G$ - change of trivialization at base point

$$\longleftrightarrow \text{map } \gamma : \Sigma \to BG / \text{homotopy}$$

$\gamma_* : \pi_1(\Sigma) \to \pi_1(BG) \cong G$

$i: * \to *$, $sk_1 \to sk_1$

Fix $\alpha \in H^3(BG, \mathbb{R}/\mathbb{Z}) \simeq H^4(BG, \mathbb{Z})$ - a group cocycle

def for Σ a closed oriented 3-mfld, set
$$Z_\Sigma := \frac{1}{|G|} \sum_{\gamma : \pi_1(\Sigma) \xrightarrow{hom} G} W_\alpha(\gamma)$$
$$W_\alpha(\gamma) = e^{2\pi i \langle \gamma^*\alpha, [\Sigma] \rangle} = e^{2\pi i \langle \alpha, \gamma_*[\Sigma] \rangle}$$
(fund class)

Rem $Z_\Sigma = $ " $\int_{Bun_G(\Sigma)} \mu_{grpd} W(\gamma)$ " , $Bun_G(\Sigma) = Hom(\pi_1(\Sigma), G)/G$ as a grpd

Rem if $\Sigma = \partial N$, N 4-dim, and γ extends over N, then $W(\gamma) = e^{2\pi i \langle \alpha, \gamma_*[\partial N] \rangle} = e^{-2\pi i \langle \delta\alpha, \tilde\gamma_*[N] \rangle} = 1$

- E.g. for $\Sigma = S^3$: $\pi_1(S^3) = \{1\}$, $\gamma : \pi_1 \to 1 \in G$ or $\Sigma \to *$, $S^3 = \Sigma$, $* \in BG$ $\Rightarrow Z_{S^3} = \frac{1}{|G|}$

- For a connected sum of 3-manifolds: $\Sigma = \Sigma_1 \# \Sigma_2$

$$\boxed{Z_{\Sigma_1 \# \Sigma_2} \cdot Z_{S^3} = Z_{\Sigma_1} \cdot Z_{\Sigma_2}}$$

Because: (1) $\pi_1(\Sigma) = \pi_1(\Sigma_1) * \pi_1(\Sigma_2)$ (take base point on ∂B^3)
free product

(2) $\langle \alpha, \gamma(\Sigma) \rangle = \langle \alpha, \gamma_1(\Sigma_1) \rangle + \langle \alpha, \gamma_2(\Sigma_2) \rangle \Leftarrow \exists$ 4-mfld N s.t. $\partial N = \Sigma \amalg \bar\Sigma_1 \amalg \bar\Sigma_2$ and extension $\tilde\gamma$ of $(\gamma, \gamma_1, \gamma_2)$ to N

short for $\gamma_*[\Sigma]$

so $0 = \langle \underset{0}{\delta\alpha}, \tilde\gamma(N) \rangle = -\langle \alpha, \tilde\gamma(\partial N) \rangle = -\langle \alpha, \gamma(\Sigma) \rangle + \langle \alpha, \gamma_1(\Sigma_1) \rangle + \langle \alpha, \gamma_2(\Sigma_2) \rangle$

- Normalization of Z_Σ is such that
$$Z_{S^2 \times S^1} = 1$$
$$" \frac{1}{|G|} \sum_{\gamma \in Hom(\underbrace{\pi_1(S^2 \times S^1)}_{\mathbb{Z}}, G)} \underbrace{W(\gamma)}_{=1} "$$
since every G-bundle over $S^2 \times S^1$ extends to $B^3 \times S^1$

• ~~Z~~ Z_Σ may vanish!

E.g. $G = \mathbb{Z}_2$, $\Sigma = \mathbb{RP}^3$

$\mathbb{RP}^3 \hookrightarrow \mathbb{RP}^\infty \rightsquigarrow [\text{image of } \mathbb{RP}^3] \in H_3(\mathbb{RP}^\infty, \mathbb{Z}) \simeq H^4(\mathbb{RP}^\infty; \mathbb{Z}) \simeq H^3(\ldots, \mathbb{R}/\mathbb{Z})$ (univ. coeff thm)

gives a class $[\alpha] \in H^3(\mathbb{RP}^3, \mathbb{R}/\mathbb{Z})$

~~Z~~ Then $Z_{\mathbb{RP}^3} = \frac{1}{2}(1 + (-1)) = \underline{\underline{0}}$

Spaces of states

phase space for a surface B: moduli space (coset) of G-bundles over B,

$$\Phi_B = \mathrm{Hom}(\pi_1(B), G)/G$$

$\tilde{\mathcal{H}}_B = \mathrm{Span}_{\mathbb{C}} \Phi_B$ – "naive" space of states

However, generally $\dim \mathcal{H}_B = Z_{S^1 \times B} \leq |\Phi_B|$

($\mathcal{H}_B$: "actual" space of states)

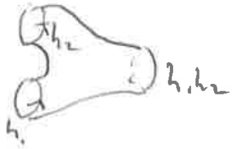
Indeed, $Z_{S^1 \times B} = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(B), G)} \sum_{h \in N_\gamma \subset G} W(\gamma, h)$

(N_γ: stabilizer of $\mathrm{Im}(\gamma)$)

• for $\alpha = 0$, $Z_{S^1 \times B} = \sum_{\gamma \in \mathrm{Hom}(\pi_1(B), G)} \frac{|N_\gamma|}{|G|} = \frac{1}{\text{~~\# of representatives of } [\gamma] \text{ in } \mathrm{Hom}(\pi_1(B)~~ \text{ (order of the orbit of } \gamma \text{ under conjugation by } G)} = |\Phi_B|$

• for α general, $W(\gamma, -)$ is a 1D-representation of N_γ,

since $W(\gamma, h_1) W(\gamma, h_2) = W(\gamma, h_1 h_2)$

since we can consider 3-manifold $B \times$ 

$$\Rightarrow Z_{S^1 \times B} = \sum_{[\gamma] \in \mathrm{Hom}(\pi_1(B), G)/G} \begin{cases} 1 & \text{if } W(\gamma, -) \text{ trivial} \\ 0 & \text{otherwise} \end{cases}$$

For manifolds with boundary, $\partial \Sigma \neq \emptyset$:

$$Z_\Sigma = \sum_{\substack{[\gamma_\partial] \in \mathrm{Hom}(\pi_1(\partial\Sigma), G)/G \\ \text{s.t. } W_{B \times S^1}(\gamma_\partial, -) = 1}} Z_\Sigma([\gamma_\partial]) \underbrace{|[\gamma_\partial]\rangle}_{\text{basis vector in } \mathcal{H}_B}$$

where $Z_\Sigma([\gamma_\partial]) := \frac{1}{|G|} \sum_{\substack{\gamma \in \mathrm{Hom}(\pi_1(\Sigma), G) \\ \text{s.t. } [\gamma|_\partial] = [\gamma_\partial]}} W(\gamma)$

Hermitian structure: $\langle [\gamma_\partial'] | [\gamma_\partial] \rangle = \begin{cases} |N_{\gamma_\partial}| = \frac{|G|}{\text{no. of repr. in } [\gamma_\partial]} & \text{if } [\gamma_\partial'] = [\gamma_\partial] \\ 0 & \text{otherwise} \end{cases}$

$\langle [\gamma_\partial'] | [\gamma_\partial] \rangle = (|\gamma_\partial'\rangle, |\gamma_\partial\rangle)$

Example / Consistency check: TQFT 3/1

Take $\Sigma = B \times [0,1]$

$$Z_\Sigma = \sum_{\substack{[\gamma_\partial] \in \mathrm{Hom}(\pi_1(B), G)/G \\ \text{s.t. } W_{B\times S^1}(\gamma_\partial, -) = 1}} \frac{1}{|G|} \cdot (\text{no of reps. in } [\gamma_\partial]) \cdot |[\gamma_\partial]\rangle \otimes |[\gamma_\partial]\rangle \in \mathcal{H}_B \otimes \mathcal{H}_B$$

$$Z_\Sigma \longmapsto \tilde{Z}_\Sigma \in \mathcal{H}_B \otimes \mathcal{H}_B^*$$

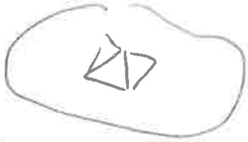
$$\mathcal{H}_B \otimes \mathcal{H}_B \xrightarrow{\mathrm{id} \otimes \langle , \rangle} \mathcal{H}_B \otimes \mathcal{H}_B^*$$

$$Z_{S^1 \times B} = \mathrm{tr}\, \tilde{Z}_\Sigma = \# \left\{ \begin{array}{l} [\gamma_\partial] \in \mathrm{Hom}(\pi_1(B), G)/G \\ \text{s.t. } W_{B\times S^1}(\gamma_\partial, -) = 1 \end{array} \right\} = \dim \mathcal{H}_B. \quad \text{Hooray!}$$

$$\frac{(\text{no of reps in } [\gamma_\partial]) \cdot |N_{\gamma_\partial}|}{|G|} = 1$$

Lattice gauge theory formulation

let T be a triangulation of Σ



$\mathrm{Bun}_G(\Sigma) :=$ {G-bundles over Σ}/iso $\simeq$ {maps g: (edges of T) $\longrightarrow G$ s.t. for a 2-simplex e_1, e_2, e_3, $g(e_3)g(e_2)g(e_1) = 1$}

convention: can change orientation of an edge $e \to \bar{e}$, changing $g(e) \longmapsto g(\bar{e}) = g(e)^{-1}$

$g\uparrow \rightsquigarrow \downarrow g^{-1}$

$G^{\text{vertices of } T}$ = maps $h: V(T) \longrightarrow G$

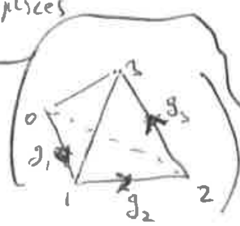
acts by $g(e) \longmapsto h_v g(e) h_u^{-1}$ for an edge $u \xrightarrow{e} v$

gauge transformations = changes of trivialization of the bundle

$$Z_\Sigma = \frac{1}{|G|^{\#\text{vertices}}} \sum_{\substack{g: \text{edges} \to G \\ \text{flat on 2-simplices,} \\ \text{i.e. } g(e_3)g(e_2)g(e_1) = 1 \\ \text{for a 2-simplex}}} \prod_{\substack{\text{3-simplices} \\ \tau}} W(\tau, \alpha)^{(\pm 1)}$$

± 1 depending on mutual orientation of τ vs. Σ

assume that all vertices have a total ordering



$W(\text{tetrahedron with vertices } 0,1,2,3 \text{ and edges } g_1, g_2, g_3) := \alpha(g_3, g_2, g_1)$

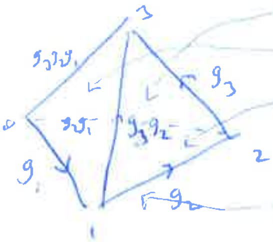
Properties (1) Does not depend on triangulation and ordering of vertices and agrees with old definition (in terms of π_1)

(2) Changing $\alpha \longmapsto \alpha + \delta\beta$ changes Z_Σ by a bdry term, $Z_\Sigma \longmapsto Z_\Sigma \times$ (something depending on $\partial\Sigma$ only)

(3) Gauge transformation $g_e \mapsto h_v g_e h_u^{-1}$ ——"——

- Change $\alpha \mapsto \alpha + \delta\beta$, $\beta \in H^2(BG)$ $\qquad \omega = e^{2\pi i \alpha(\cdot,\cdot,\cdot)} \qquad B = e^{2\pi i \beta(\cdot,\cdot)}$

$$\omega(T; g_3,g_2,g_1) \longmapsto \omega(g_3,g_2,g_1)\cdot \frac{B(g_3, g_2 g_1)\, B(g_2,g_1)}{B(g_3,g_2)\, B(g_3 g_2, g_1)}$$



- change $\underset{\text{1-simplex}}{g_{(ij)}} \longmapsto h_j^{-1}\, g_{(ij)}\, h_i$ $\qquad$ take h_0 non-triv, $h_{\neq 0} = 1$:

$$\omega(T; g_3,g_2,g_1) \longmapsto \omega(T, g_3,g_2,g_1)\cdot \frac{\omega(h,g_3,g_2)\, \omega(h, g_3 g_2, g_1)}{\omega(h, g_3, g_2 g_1)}$$

Remarks

- mapping class group of B, $MCG(B) = \pi_0(\mathrm{Diff}(B))$ acts on Φ_B by permutations, and hence acts on $\mathcal{H}_B$ linearly.

e.g. $\Phi_{S^1_t \times S^1_t} = \{(a,b) \in G\times G \mid ~~[a,b]=1~~\ aba^{-1}b^{-1}=1\} / (a,b)\sim(hah^{-1}, hbh^{-1})\ \forall h\in G$

$T: \quad (a,b) \to (a, a^{-1}ba^{-1})$

(?) $S: \quad (a,b) \to (b^{-1}, a)$

- case $\dim \Sigma = 2$, $d=0$: counting G-bundles on surfaces:

$$\mathcal{H}_{S^1} = \mathrm{Span}_{\mathbb{C}} \underbrace{G/G}_{\text{set of conjugacy classes}}$$

$$Z_\Sigma(\underbrace{A_1,\dots,A_n}_{\in G/G}) = \frac{1}{|G|}\sum_{\substack{\gamma:\pi_1(\Sigma)\xrightarrow{\text{hom}} G \\ \gamma|_{\underbrace{\partial_i\Sigma}_{i\text{-th circle}}} = A_i}} 1$$

$$\langle A \mid B\rangle = \begin{cases} |N_A| & \text{if } A = B^{-1} \\ 0 & \text{otherwise}\end{cases} \qquad A, B \in G/G$$

another convenient basis (of the dual): characters of irr. reps of G

$\{R\}$ - irreps of G, $\rho_R: G \to \mathbb{C}$ - invariant under conjugation, $\rho_R \in \mathrm{Fun}(G)^G = \mathrm{Fun}(G/G)$

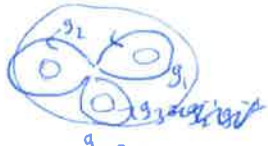
- trace of group element in repr. R

- ~~pairing between $\mathrm{Span}\{R\}$ and $\mathrm{Span}(G/G)$ is non-degenerate~~ given by ~~pairing~~ $\langle R \mid A\rangle = \rho_R(A)$ go to the back-side!

$\{\langle R|\}$ - basis of $\mathcal{H}^*_{S^1}$

for a pair of pants, $\left(\langle R_1| \otimes \langle R_2| \otimes \langle R_3|, Z\left(\right.\right.$  $\left.\left.\right) ~~x~~ \right) =$

$$= \frac{1}{|G|}\sum_{g_1,g_2\in G} \rho_{R_1}(g_1)\rho_{R_2}(g_2)\rho_{R_3}(g_1^{-1}g_2^{-1}) \ (=)$$



$g_3 g_2 g_1 = 1$

Orthogonality of characters: ~~$\sum_{g\in G}\rho_R(g)\rho_{R'}(g^{-1}h) = \begin{cases}|G| \frac{\rho_R(h)}{\dim R}, & \text{if } R'=R \\ 0 & \text{otherwise}\end{cases}$~~

$$(=) \ \frac{1}{|G|}\sum_{g_2\in G} |G|\, \delta_{R_1,R_3}\, \rho_{R_2}(g_2)\rho_{R_1}(g_2^{-1}) = \begin{cases} |G|\cdot \dim R & \text{if } R_1=R_2=R_3 \\ 0 & \text{otherwise}\end{cases}$$

$$\sum_{A\in G/G} \frac{1}{|N_A|} |A\rangle \otimes |A^{-1}\rangle$$

cylinder:  : $\left(\langle R_1| \otimes \langle R_2|, Z\left(\right.\right.$  $\left.\left.\right)\right) = \frac{1}{|G|}\sum_{g\in G}\rho_{R_1}(g)\rho_{R_2}(g^{-1}) =$

$$= \begin{cases} \dim(R) & \text{if } R_1 = R_2 \\ 0 & \text{otherwise}\end{cases}$$

let $|A\rangle = \sum_R \rho_R(A)\ |R\rangle$ $\qquad = \frac{1}{|G|}\sum_{g\in G} |[g]\rangle \otimes |[g^{-1}]\rangle$ $\quad =$ by Schur's orthogonality

then $Z\Big($  $\Big) = \sum_R \frac{1}{|G|} \sum_{A\in G/G} \frac{|G|}{|N_A|} \cdot |A\rangle \otimes |A^{-1}\rangle = \sum_R \cancel{d_R\, R} \cdot |R\rangle \otimes |R\rangle$

no of representatives of A in G

$(|R\rangle, |R'\rangle) = \begin{cases} \cancel{\ \ } \ \text{if } R = R' \ (\text{or } R'^* \,??) \\ 0 \ \text{otherwise} \end{cases}$ $\quad \Big|\ \Rightarrow Z(S^1\times S^1) = \#\{R\} = |G/G| = \dim \mathcal{H}_{S^1}$

$Z\Big($  $\Big) = \frac{1}{|G|}\ \underset{\widehat{G/G}}{|[1]\rangle} = \frac{1}{|G|}\sum_R \dim R \cdot |R\rangle$ $\quad \Big|\ \Rightarrow Z(S^2) = \frac{1}{|G|^2}\sum_R (\dim R)^2 = \frac{1}{|G|}$

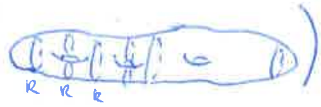
$\sum_{g\in G} \rho_R(g)\, \underbrace{\overline{\rho_{R'}(g)}}_{\rho_{R'^*}(g)\ \overset{?}{=}\ \rho_R(g^{-1})} = |G|\,\delta_{RR'}$ $\ (*) \rightsquigarrow$ stronger version: $\sum_{g\in G} \rho_R(g)\,\rho_{R'}(g^{-1}h) = \begin{cases} \frac{|G|}{\dim R}\,\rho_R(h) & \text{if } R = R' \\ 0 & \text{otherwise} \end{cases}$

$\sum_R \chi_R(g)\,\overline{\chi_R(h)} = \begin{cases} |N_g| & \text{if } g, h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$ $\qquad \left(\Rightarrow \sum_R (\dim R)^2 = |G| \right)$

$Z\Big($  $\Big) = \frac{1}{|G|} \sum_{g_1,g_2\in G} |[g_1]\rangle \otimes |[g_2]\rangle \otimes |[g_1^{-1}g_2^{-1}]\rangle =$

$= \frac{1}{|G|} \sum_{R_1,R_2,R_3} \Big(\sum_{g_1,g_2\in G} \rho_{R_1}(g_1)\,\underbrace{\rho_{R_2}(g_2)\,\rho_{R_3}(g^{-1}g_1^{-1})}_{\delta_{R_2R_3}\frac{|G|}{\dim R_2}\rho_{R_2}(g_1^{-1})} \Big)\ |R_1\rangle \otimes |R_2\rangle \otimes |R_3\rangle =$

$= |G| \sum_R \frac{1}{\dim R}\ |R\rangle \otimes |R\rangle \otimes |R\rangle$

$Z\Big($  $\Big) = Z\Big($  $\Big) = |G|^{2g-2} \sum_R (\dim R)^{2-2g}$

genus g

surface with n bdry circles: $Z\Big($  $\Big) = \sum_R \left(\frac{\dim R}{|G|}\right)^{2-2g-n} |R\rangle \otimes \dots \otimes |R\rangle \ \in (\mathcal{H}_{S^1})^{\otimes n}$

* from $|R\rangle$ to $|A\rangle$: $\quad |R\rangle = \sum_{\{A\}\in G/G} \frac{1}{|N_A|}\,\rho_R(A)\,|A\rangle$

Yang-Mills theory

(M,g) - Riemannian (Pseudo-Riemannian) manifold, $\dim M = n$

G - Lie group

Space of fields: $F = \mathrm{Conn}\left(\begin{smallmatrix} G\times M \\ \downarrow \\ M\end{smallmatrix}\right) \simeq \underset{A \,\in}{\overset{\text{"}Lie(G)}{\mathfrak{g}}} \otimes \Omega^1(M)$

action: $S(A) = \mathrm{tr}\int_M \frac{1}{2e^2} F_A \wedge * F_A$

$F_A = dA + A\wedge A$ - curvature

Hodge star assoc. to g

e - "charge of the gluon" - coupling constant

eq. of motion: $d_A * F_A = 0$ - Yang-Mills equation

Gauge symmetry: $A \mapsto h^{-1}Ah + h^{-1}dh$
$h: M \to G$

Rem: case $G = \mathbb{R}$ (or $U(1)$) - classical electrodynamics, Yang-Mills eq. $\rightsquigarrow$ Maxwell's eq.

First order ~~reformulation~~ formalism

$$F = \underset{A}{\mathfrak{g}\otimes\Omega^1(M)} \oplus \underset{B}{\mathfrak{g}\otimes\Omega^{n-2}(M)}$$

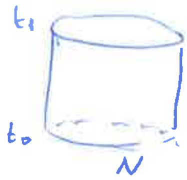
$$S(A,B) = \mathrm{tr}\int_M B\wedge F_A + \frac{e^2}{2} B\wedge * B$$

Equations of motion: $\begin{cases} F_A + e^2 * B = 0 \\ d_A B = 0 \end{cases}$

gauge sym: $A \mapsto h^{-1}Ah + h^{-1}dh$
$B \mapsto h^{-1}Bh$

Hamiltonian formalism

Take $M = [t_0,t_1]\times N$ (or more generally, $\partial M = N$)



phase space: $\Phi_{N,t}$ = restrictions of fields to N = $= \mathfrak{g}\otimes\Omega^1(N) \oplus \mathfrak{g}\otimes\Omega^{n-2}(N)$

variation of action

$$\delta S = \mathrm{tr}\int_M \delta B\wedge(F_A + e^2 * B) + B\wedge(d\,\delta A + \delta A\wedge A + A\wedge\delta A) = \text{Stokes' thm}$$

$$= \mathrm{tr}\int_M\Big(\delta B\wedge(F_A + e^2 * B) + \underbrace{(-1)^n d_A B\wedge\delta A}_{=\pm\,\delta A\wedge d_A B}\Big) + (-1)^n\,\mathrm{tr}\int_{\partial M} B\wedge\delta A$$

$$\alpha_N = \mathrm{tr}\int_N B_N\wedge\delta A_N \in \Omega^1(\Phi_N)$$

Symplectic structure $\omega_N = \delta\alpha_N = \mathrm{tr}\int_N \delta B_N\wedge\delta A_N \in \Omega^2(\Phi_N)$ ~~symplectic str.~~ symp. form on Φ_N

$\underset{\text{coiso}}{\Phi_N \supset C_N} = \{(A_N,B_N)\in\Phi_N \mid \exists$ sol. of e. of m. $(\tilde A,\tilde B)$ on $N\times[0,\varepsilon]$ restricting to (A_N,B_N) on $N\times\{0\}\}$

"space of allowed Cauchy data",

char. distribution $(\ker \omega_N|_{C_N})_{(A,B)} \subset T_{(A,B)}\Phi_N$ = gauge transformations with t-indep. generator

Hamiltonian: $H = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon} S(\tilde A,\tilde B) \in C^\infty(C_N)^{\text{char. distr.}}$

Evolution: Flow of Ham. vector field $\{H,\cdot\}$, i.e. $\frac{\partial}{\partial t}A_{N,t} = \{H, A_{N,t}\}_{\omega_N}$
same for $B_{N,t}$

In Yang-Mills:

$C_N = \{(A_N, B_N) \mid d_{A_N} B_N = 0\}$ "Gauss law" constraint

$\underline{C} = C/_{A_N \sim h^{-1}A_N h + h^{-1}dh,\ B_N \sim h^{-1} B_N h} \simeq T^*\left(\mathrm{Conn}(N)/_\sim\right)$

$H = \mathrm{tr}\int_N \frac{1}{2e^2} \underbrace{F_{A_N}}_{} \wedge * F_{A_N} + \frac{e^2}{2} \underbrace{B_N}_{} \wedge * B_N$

in Maxwell theory: "magnetic field" "electric field"

2D Yang-Mills

$F_\Sigma = \mathfrak{g}\otimes\Omega^1(\Sigma) \oplus \mathfrak{g}\otimes\Omega^0(\Sigma)$, $S = \mathrm{tr}\int_\Sigma B F_A + \frac{e^2}{2}\mu B^2$

surface

E. of M.: $F_A = -e^2\mu\cdot B$, $d_A B = 0$

$*1 =$ (metric) volume element

Note: only $\mu = \sqrt{\det g}\, d^2x$ enters S, not the whole metric $\Rightarrow$

$\Rightarrow$ invariance under area-preserving diffeo (instead of isometries)

$\Phi_{S^1} = \mathfrak{g}\otimes\Omega^1(S^1)\oplus\mathfrak{g}\otimes\Omega^0(S^1)$

$\cup$

$C \quad d_A B = 0$ $\qquad \underline{C} = T^*\left(\mathrm{Conn}(S^1)/_\sim\right) = T^*(G/_G)$

$H = \frac{e^2}{2}\,\mathrm{tr}\oint_{S^1}\mu_{S^1} B^2 = \frac{e^2 L}{2}\,\mathrm{tr}\, B(0)^2$

fixed point on S^1

Can. quantization: $\mathcal{H}_{S^1} = L^2(G)^G$

$\mathcal{H}^{big}_{S^1} = \text{"}L^2\text{"}(\mathfrak{g}\otimes\Omega^1(S^1))$

- functions of A_x

G-invariant functions of $W = P\exp\oint_{S^1} A_x dx$

basis in $\mathcal{H}_{S^1}$ given by characters of irreps, $\rho_R(W)$

$\hat{H} = \frac{e^2}{2}\,\mathrm{tr}\oint_{S^1} dx \frac{\delta}{\delta A_x^a(x)}\frac{\delta}{\delta A_x^a(x)}$, $\hat{H}\rho_R(W) = \frac{e^2}{2} L\, C_2(R)\cdot\rho_R(W)$

"$\frac{e^2}{2}\oint_{S^1} dx\, \rho_R(T^a T^a W)$"

TQFT 5/1

20.03.13

Yang-Mills theory

Reminder • 2nd order formalism: $F = \mathfrak{g}\otimes\Omega^1(M) \ni A$, $S(A) = \frac{1}{2e^2}\mathrm{tr}\int_M F_A \wedge * F_A$

$F_A = dA + A\wedge A$

gauge transformations: $A \mapsto h^{-1}Ah + h^{-1}dh$

$h: M \to G$

EL eq: $\boxed{d_A * F_A = 0}$

• 1st order formalism: $F = \underset{A}{\mathfrak{g}\otimes\Omega^1(M)} \oplus \underset{B}{\mathfrak{g}\otimes\Omega^{n-2}(M)}$, $S(A,B) = \mathrm{tr}\int_M B\wedge F_A - \frac{e^2}{2}B\wedge * B$

E-L eq: $F_A - e^2 * B = 0$

$d_A B = 0$

gauge transf.: $A \mapsto h^{-1}Ah + h^{-1}dh$

$B \mapsto h^{-1}Bh$

Ref: arXiv:1207.0239

Aside Class. gauge theories:

Lagrangian formalism

M - spacetime mfld

F_M - space of fields - space of sections of a sheaf over M

$S_{M,g} = \int_M \mathcal{L}(\underbrace{\varphi, \partial\varphi, \partial^2\varphi, \dots}_{\text{finitely many derivatives}}; g)$ - action

g - geometric data on M

Dynamics: Euler-Lagrange equations for S

Gauge symmetry: distribution on F, preserving S

Hamiltonian formalism

$M = N\times[t_0, t_1]$ - cylinder

(Φ_N, ω) - phase space (symplectic)

$\cup$ coisotropic

C "constraint surface"

(char. distrib. on C) = gauge transf. with t-independent parameter

$\downarrow$ reduction

$(\underline{C}, \underline{\omega})$ - reduced phase space

Dynamics: flow of a Hamiltonian vector field

$\frac{\partial}{\partial t}\varphi = \{H, \varphi\}$, $H \in C^\infty(\underline{C})$

Construction: assuming S is 1st order in derivatives of φ,

(a) $\tilde{\Phi}_N = \{$restrictions of fields $\varphi \in F_{N\times[0,\varepsilon]}$ to $N\times\{0\}\}$ - pre-phase space

$\tilde{\alpha}_N = $ boundary term of $\delta S_{N\times[0,\varepsilon]}$, $\tilde{\omega}_N := \delta\tilde{\alpha}_N \in \Omega^2(\tilde{\Phi}_N)$ - (pre)symplectic form

$\tilde{\alpha}_N \in \Omega^1(\tilde{\Phi}_N)$

(b) $\Phi_N := \tilde{\Phi}_N/\ker\tilde{\omega}_N$ - presymplectic reduction

$\Phi \supset C$: $\{$restrictions of sol. of EL eq. on $N\times[0,\varepsilon]$ to $N\times\{0\}\}/\ker\tilde{\omega}_N$ = space of Cauchy data for EL equations

coisotropic*

(c) $\Phi_N^{red} := \underline{C}$

(d) Hamiltonian

consider $L_{N\times[0,\varepsilon]} \subset \Phi_N^{red}\times\Phi_N^{red}$ = graph $(U_\varepsilon : \Phi_N^{red} \to \Phi_N^{red})$

$= \{$restrictions of sol. on $N\times[0,\varepsilon]$ to $N\times\{0\}$ and $N\times\{\varepsilon\}\}/\ker\tilde{\omega}$

L is isotropic

(since $\tilde{\alpha}_N|_\Sigma = \delta S|_{EL_N}$; bulk term vanishes on EL_N

$\Rightarrow \delta\tilde{\alpha}_N|_\Sigma = \delta^2 S|_{EL_N} = 0$)

$\Rightarrow U_\varepsilon$ symplectomorphism $\xrightarrow{\text{semi-group law } U_{\varepsilon+\varepsilon'} = U_\varepsilon\circ U_{\varepsilon'}}$ U_ε = flow of a symplectic v. field X

$\xrightarrow{\text{assuming obstruction vanishes}}$ $X = \{H, \cdot\}$, $H \in C^\infty(\underline{C})$

* - assumption? (that constraints form a Poisson subalgebra)

Rem • if S is not 1^{st} order, set $\tilde{\Phi}_N$ = values of fields on $N\times\{t\}$ and first k derivatives in t-direction.

• Other constructions of H – via Legendre transform of $L = \int_{N\times\{t\}} \mathcal{L}$

– via stress-energy tensor $T = \frac{\delta S}{\delta g}$, $H = \int_{N\times\{t\}} T_{tN}$

Yang-Mills (1^{st} order)

$M = N \times [t_0, t_1]$

$$\tilde{\Phi}_N = \Phi_N = \{(A_N, B_N) \in \mathfrak{g}\otimes\Omega^1(N) \oplus \mathfrak{g}\otimes\Omega^{n-2}(N)\}$$

$$\tilde{\alpha}_N = \alpha_N = (-1)^{n} \mathrm{tr}\int_N B \wedge \delta A \quad , \quad \tilde{\omega}_N = \omega_N = -\mathrm{tr}\int_N \delta B \wedge \delta A$$

• $\Phi_N \supset C = \{(A_N, B_N) \mid d_{A_N} B_N = 0\}$ • $\underline{C} \simeq T^*\left(\mathrm{Conn}(N)/\sim\right)$

EL equations $\begin{cases} F_A = e^2 * B \\ d_A B = 0 \end{cases}$ $\overset{\text{split}\ A = \alpha dt + A_N,\ B = dt\,\beta + B_N}{\Longleftrightarrow}$

$$\begin{cases} e^2 \underbrace{* B_N}_{dt\,(*_N B_N)} = (F_A)_t = d_N(\alpha dt) + d_t A_N + [\alpha dt, A_N] \\ e^2 \underbrace{*(dt\,\beta)}_{*_N\beta} = (F_A)_N = d_N A_N + A_N \wedge A_N \\ d_{A_N} B_N = 0 \\ d_t B_N + [\alpha dt, B_N] + d_N(dt \cdot \beta) + [A_N, dt\,\beta] = 0 \end{cases}$$

imposing gauge $\alpha = 0$

$$\Longleftrightarrow \begin{cases} \partial_t A_N = e^2(*_N B_N) \\ F_{A_N} = e^2 *_N \beta \\ d_{A_N} B_N = 0 \\ \partial_t B_N = d_{A_N}\beta \end{cases} \underset{\text{eliminate } \beta}{\Longleftrightarrow} \begin{cases} (1)\ d_{A_N} B_N = 0 \\ (2)\ \partial_t A_N = e^2(*_N B_N) \\ (3)\ \partial_t B_N = \frac{1}{e^2} d_{A_N} *_N F_{A_N} \end{cases}$$

(1) – constraint; (2), (3) define time evolution for (A_N, B_N)

(2) + (3) – Hamiltonian flow equations for $\boxed{H = \mathrm{tr}\int_N \frac{e^2}{2} \underbrace{B_N}_{\text{"electric field"}} \wedge *_N B_N + \frac{1}{2e^2} F_{A_N} \wedge *_N \underbrace{F_{A_N}}_{\text{"magnetic field"}}}$

$\in C^\infty(C)^{\text{char. distrib.}} = C^\infty(\underline{C})$

$$\left(\begin{array}{c}\text{char. distribution} \\ \ker(\omega_N|_C) \subset T_{(A_N,B_N)} C\end{array}\right) = \left(\begin{array}{c}\text{gauge transf.} \\ \text{with } t\text{-indep.} \\ \text{generator}\end{array}\right) = \left\{ \begin{array}{l} A_N \mapsto h^{-1} A_N h + h^{-1} d_N h \\ B_N \mapsto h^{-1} B_N h \end{array} \,\middle|\, h: \underset{\sim}{N} \to G \right\}$$

2D Yang-Mills   Ref A.A. Migdal "Recursion relations in gauge theories", 1975

Σ - surface $F_\Sigma = \mathfrak{g}\otimes\Omega^1(\Sigma) \oplus \underset{\substack{\Psi\\ B}}{\mathfrak{g}\otimes\Omega^0(\Sigma)}$ ($A \in \mathfrak{g}\otimes\Omega^1(\Sigma)$)

$$\boxed{S = \mathrm{tr}\int_\Sigma B F_A - \frac{e^2}{2}\mu B^2}$$

μ: volume element, $\mu = *1$

Note: S depends on the vol. elt $\mu = \sqrt{\det g}\, d^2x$, not on the whole metric
$\Rightarrow$ invariance under area-preserving diffeo (instead of isometries)

EL eq.: $\begin{cases} F_A = e^2 B\mu \\ d_A B = 0 \end{cases}$

$\Phi_{S^1} = \mathfrak{g}\otimes\Omega^1(S^1)\oplus\mathfrak{g}\otimes\Omega^0(S^1)$
$\cup$
$C \quad d_A B = 0$

$$\underline{C} = T^*(\mathrm{Conn}(S^1)/\sim) = T^*(G/G)$$

$$H = \frac{e^2}{2}\,\mathrm{tr}\oint_{S^1}\mu_{S^1} B^2 \underset{\substack{\text{using}\\ \text{constant}}}{=} \frac{e^2 L}{2}\,\mathrm{tr}\, B(0)^2$$

0: fixed point on S^1

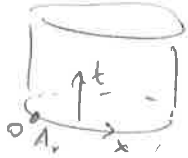
(Canonical) Quantization:

$\mathcal{H}_{S^1} = L^2(G)^G$ $\rightsquigarrow$ $= \{G\text{-invariant functions of } W = P\exp\oint A_x dx\}$

$\mathcal{H}^{b.g}_{S^1} = \mathrm{Fun}(\mathfrak{g}\otimes\Omega^1(S^1))$
"L^2"

- Hermitian structure:
$\langle f|g\rangle = \int_G x\, \bar{f} g$

x: Haar measure, normalized by $\mathrm{Vol}(G) = 1$.



Peter-Weyl thm

For G compact, $L^2(G) = \bigoplus_{\substack{R\in\mathrm{Irrep}(G)\\ \text{unitary irreducible}}} R\otimes R^*$

basis in $L^2(G)$ is given by matrix elements of irreps R.

Basis $|R\rangle$ in $L^2(G)^G$ - characters of irreps, ~~given~~ $\chi_R = \mathrm{Tr}_R(\hat{G})$

Quantized Hamiltonian:

$B^a(x) \mapsto \frac{\delta}{\delta A^a(x)}$, $\quad H \longmapsto \hat{H} = \frac{e^2 L}{2}\frac{\delta}{\delta A^a_x(0)}\frac{\delta}{\delta A^a_x(0)}$

a: corresp to o/n basis in $\mathfrak{g}$

In rep. basis:

$\hat{H}|R\rangle = \hat{H}\,\mathrm{tr}_R P\exp\oint A_x dx = \frac{e^2 L}{2}\mathrm{tr}_R \underbrace{T^a T^a}_{C_2(R)\mathrm{Id}} P\exp\oint A_x dx =$

$C_2(R)$ — quadratic Casimir

$= \frac{e^2 L}{2} C_2(R)\,|R\rangle$

Evolution operator for cylinder: $Z_{S^1\times[0,T]} = e^{-\frac{i}{\hbar}\hat{H}T} = \sum_R e^{-a C_2(R)} |R\rangle\langle R| \in \mathrm{End}(\mathcal{H}_{S^1})$

$a = \frac{i}{\hbar}\frac{e^2}{2} LT$

$|R\rangle\langle R|$: projector to $\mathrm{Span}(|R\rangle)$

or $Z_{S^1\times[0,T]}(g_0, g_1) = \sum_R \chi_R(g_0^{-1})\,\chi_R(g_1)\, e^{-\frac{i}{\hbar}\frac{e^2}{2}LT\, C_2(R)}$

Thm (Peter Weyl):

For G compact

- matrix coefficients of irreps are dense in $L^2(G)$
- unitary representations are completely reducible (into fin.dim unitary irreducible reps)
- $L^2(G) \underset{\text{reg.rep}}{\simeq} \bigoplus R^{\oplus \dim R}$

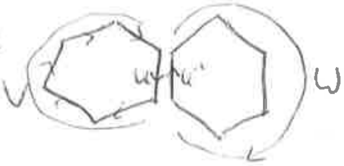
Disk:

area $\to 0$ $\Rightarrow$ $Z_{\text{disk}}(W)$ = quantization of Lagrangian submfld $\{W=1\} \subset T^*(G/_G)$ $= \delta(W,1)$ ← wrt Haar measure

finite area attach a cylinder

$$Z_{\text{disk}}(a,W) = \sum_R \dim R \; \chi_R(W)\, e^{-a\, C_2(R)}$$

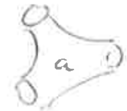
Gluing:



$$Z(\underset{a=a_1+a_2}{\text{glued domain}}, VW) = \int_G dU \; Z_{\text{disk}}(a_1, VU)\, Z_{\text{disk}}(a_2, U^{-1}W)$$

$\to$ can calculate Z for any surface

- pair of pants:



$$Z(a; W_1, W_2, W_3) = \sum_R \frac{\chi_R(W_1)\chi_R(W_2)\chi_R(W_3)}{\dim R}\, e^{-a\, C_2(R)}$$

closed surface of genus h

$$Z = \sum_R (\dim R)^{2-2h}\, e^{-a\, C_2(R)}$$

Orthogonality relations:
$$\int_G dU \; \chi_{R_1}(VU)\chi_{R_2}(U^{-1}W) = \delta_{R_1 R_2} \frac{\chi_{R_1}(VW)}{\dim R_1}$$

Note: in "topological sector" $a=0$, $Z(S^2)$ and $Z(S^1\times S^1)$ diverge.

Rem: can consider deformations of Sym_2 by G-inv. polynomials in B
$\rightsquigarrow$ $\hat{H}$ gets deformed by higher Casimirs

Turaev - Viro invariants

Ref: V. G. Turaev, O. Ya. Viro, "State-sum invariants of 3-manifolds and quantum 6j-symbols"

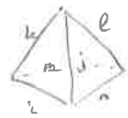
Topology 31, 4 (1992) 865-902

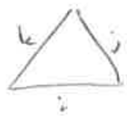
fix (M, X) - compact 3-mfd with triangulation

$r \in \mathbb{N}$, $r \geq 3$ q - complex, primitive root of unity of degree r.

(*) $|M| = \sum\limits_{\substack{\varphi\text{-admissible decoration}\\ \text{of 2-simplices}}} |M|_\varphi$, $|M|_\varphi = w^{-2(\#\text{vertices})} \prod\limits_{\text{edges}} w^2_{\varphi(E)} \prod\limits_{\substack{\text{3-simplices}\\ T}} |T|_\varphi$

$\varphi : \{E_1, \dots, E_b\} \to \underbrace{\{0, \tfrac{1}{2}, \dots, \tfrac{r-2}{2}\}}_{I \text{ - colors}}$





"admissible" if $i, j, k \in I$

- $i+j+k \in \mathbb{Z}$
- $i+j \geq k$, $i+k \geq j$, $j+k \geq i$
- $i+j+k \leq r-2$

$q_0 = q^{1/2} \left(= e^{\frac{\pi i}{r}}\right)$

notation: for $n \geq 1$, $[n] = \dfrac{q_0^n - q_0^{-n}}{q_0 - q_0^{-1}} \in \mathbb{R}$

- $[n]! = [n][n-1] \cdots [2][1]$
- $\Delta(i,j,k) = \left(\dfrac{[i+j-k]!\,[i+k-j]!\,[j+k-i]!}{[i+j+k+1]!} \right)^{1/2}$

- Rakah-Wigner symbol: $\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW} = \Delta(i,j,k)\,\Delta(i,m,n)\,\Delta(j,l,n)\,\Delta(k,l,m) \times$

$\times \sum\limits_{\substack{z \geq 0 \\ \text{s.t. all expressions in } [\cdot\cdot] \\ \text{are} \geq 0}} (-1)^z\, [z+1]! \Big([z-i-j-k]!\,[z-i-m-n]!\,[z-j-l-n]!\,[z-k-l-m]! \cdot [i+j+l+m-z]!\,[i+k+l+n-z]!\,[j+k+m+n-z]! \Big)^{-1}$

- 6j-symbol: $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = (\sqrt{-1})^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW}$

- $w = \dfrac{\sqrt{2r}}{|q_0 - q_0^{-1}|}$ - $w_i = (\sqrt{-1})^{2i}\,[2i+1]^{1/2}$

(*)

- Extends to manifolds with boundary, $\widetilde{\mathcal{H}}_{\Sigma, Y} = \mathrm{Span}_{\mathbb{C}} \{\text{admissible colorings of } Y\}$ (Y - triangulation)
- $\widetilde{Z}_M : \widetilde{\mathcal{H}}_{\Sigma_{in}} \to \widetilde{\mathcal{H}}_{\Sigma_{out}}$ given by colorings with fixed col. on bdry.
- $\mathcal{H}_\Sigma := \mathrm{Coim}\left(\widetilde{Z}_{\Sigma \times [0,1]} \; \widetilde{\mathcal{H}}_\Sigma \to \widetilde{\mathcal{H}}_\Sigma \right)$ $\rightarrow$ non-oriented TQFT

Independence on triangulation X $\Leftarrow$ identities for 6j-symbols, implying consistency with Pachner moves

"Biedenharn - Elliott identity"

extension of
TV invariants to a TQFT.

(1) $\tilde{\mathcal{H}}_{\Sigma, Y} := \mathrm{Span}_{\mathbb{C}}\{\text{admissible colorings of } Y\}$

surface — triangulation of Σ

(2) $\tilde{Z}_{M}$ for $\partial M = \Sigma_{in} \amalg \Sigma_{out}$, X - triang. of M
M - mfd with bdry — restricting to $Y_{in,out}$ on $\Sigma_{in,out}$

$\tilde{Z}_{M,X} : \tilde{\mathcal{H}}_{\Sigma_{in}, Y_{in}} \to \tilde{\mathcal{H}}_{\Sigma_{out}, Y_{out}}$ given by formula (*) with fixed colorings on $Y_{in,out}$.

Relative version of Pachner's theorem $\Rightarrow \tilde{Z}_{M,X}$ depends on $Y_{in,out}$ only, not on details of triang X in the bulk of M.

(3) for Y, Y' two triangulations of Σ,
any triang. X of $\Sigma \times [0,1]$ restricting to Y, Y' on $\Sigma\times\{0\}, \Sigma\times\{1\}$ gives a canonical (by (2)) map $\tilde{Z}_{\Sigma\times[0,1], X} : \tilde{\mathcal{H}}_{\Sigma, Y} \to \tilde{\mathcal{H}}_{\Sigma, Y'}$ (**)

(4) $(\tilde{\mathcal{H}}, \tilde{Z})$ defines a <u>semi-functor</u> $\mathrm{Triang.Cob}_3 \to \mathrm{Vect}_{\mathbb{C}}$; $\Sigma\times[0,1]$ cylinders is <u>not</u> mapped to an identity $id_{\tilde{\mathcal{H}}_\Sigma}$, but to a projector.

Remedy: set
(a) $\mathcal{H}_{\Sigma, Y} := \mathrm{Coim}\left(\tilde{Z}_{\Sigma\times[0,1], X} : \tilde{\mathcal{H}}_{\Sigma,Y} \to \tilde{\mathcal{H}}_{\Sigma,Y}\right)$
Coim = domain/ker — extension of $Y\times\{0\}\amalg Y\times\{1\}$ into the bulk, for some triang Y of Σ

(b) for general M, $\tilde{Z}_{M,X}$ induces homo
$Z_{M,X} : \mathcal{H}_{\Sigma_{in}, Y_{in}} \to \mathcal{H}_{\Sigma_{out}, Y_{out}}$

(c) $Z_{\Sigma\times[0,1], X} : \mathcal{H}_{\Sigma, Y} \xrightarrow{\sim} \mathcal{H}_{\Sigma, Y'}$ is a canonical isomorphism by (3)
So, we have a triang.-independent space of states $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma, Y}$ $\forall Y$

(d) Thus, by (2), $Z_{M,X} = Z_M$

(e) $Z_{\Sigma\times[0,1]} = id_{\mathcal{H}_\Sigma}$

(5) for $\varphi : \Sigma \to \Sigma'$ ~~diffeo~~ homeomorphism, set $M_\varphi = \Sigma\times[0,1]/_{\{x,1\}\sim\varphi(x)}$, $\partial M_\varphi = \Sigma \amalg \Sigma'$ "mapping cylinder"

$\varphi^\# := Z_{M_\varphi} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$

- $\varphi^\#$ only depends on isotopy class of φ and defines the action of Mod_Σ (modular group) on $\mathcal{H}_\Sigma$.

(Turaev pp 358-359)

(6) $(\mathcal{H}, Z, \varphi_\#)$ defines a <u>non-oriented</u> Atiyah's 3-TQFT.

Wigner's
6j-symbols ($\underset{\text{up to sign}}{=}$ Racah's W-coefficients)

For group $SU(2)$: $\{V_j\}_{j=0,1/2,1,3/2,\dots}$ - fin.dim. irreducible representations.

V_j comes with a standard basis $\{|j,m\rangle\}_{m=-j,-j+1,\dots,j-1,j}$ s.t. $H|j,m\rangle = m|j,m\rangle$

$\dim V_j = 2j+1$ $\qquad E|j,m\rangle \propto |j,m+1\rangle$

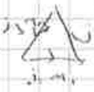
$F|j,m\rangle \propto |j,m-1\rangle$

$$V_{j_1}\otimes V_{j_2} \simeq \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} V_J$$

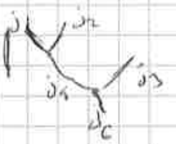
$$|(j_1 j_2) J M\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \underbrace{\langle j_1 m_1\, j_2 m_2 | J M\rangle}_{\text{Clebsch-Gordan coefficients}}$$

Wigner 3j-symbol:

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\langle j_1 m_1 j_2 m_2 | j_3 -m_3\rangle$$



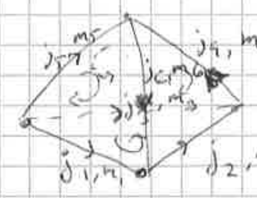
- $(V_{j_1}\otimes V_{j_2})\otimes V_{j_3} = V_{j_1}\otimes(V_{j_2}\otimes V_{j_3})$ - two bases



$M \in \{-j_6,\dots,j_6\}$

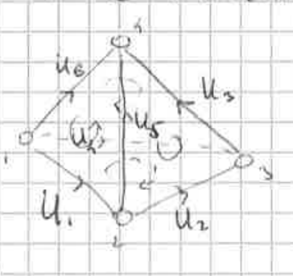
6j-symbols

- $$\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix} = \sum_{m_1\dots m_6} (-1)^{\sum_1^6 (j_k - m_k)} \begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & -m_3\end{pmatrix}\begin{pmatrix} j_1 & j_5 & j_6\\ -m_1 & m_5 & m_6\end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_3\\ m_4 & -m_5 & m_3\end{pmatrix}\begin{pmatrix} j_4 & j_2 & j_6\\ -m_4 & -m_2 & -m_6\end{pmatrix}$$

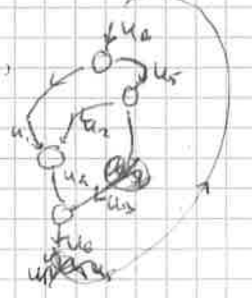


- tetrahedral symmetry:
sym. group S_4 acts by permuting vertices of tetrahedron $\to$
$\to$ permutes edges ; 6j-symbol is $S_4^{?}$-invariant

- more general / invariant viewpoint:
for $\mathcal{C}$ a semi-simple abelian tensor category with simple objects $\{U_k\}$,



$$\begin{Bmatrix} U_1 & U_2 & U_4\\ U_3 & U_6 & U_5\end{Bmatrix} \in \left[\mathrm{Hom}(U_1\otimes U_5, U_6)\otimes \mathrm{Hom}(U_2\otimes U_3, U_5)\otimes \otimes \mathrm{Hom}(U_4, U_1\otimes U_2)\otimes \mathrm{Hom}(U_6, U_4\otimes U_3)\right]$$



or $\bigoplus_{U_4}\mathrm{Hom}(U_1\otimes U_2, U_4)\otimes \mathrm{Hom}(U_4\otimes U_3, U_6) \xrightarrow{\sim} \bigoplus_{U_5}\mathrm{Hom}(U_1\otimes U_5, U_6)\otimes \mathrm{Hom}(U_2\otimes U_3, U_5)$

$\searrow\sim\qquad \mathrm{Hom}(U_1\otimes U_2\otimes U_3, U_6) \nearrow$

Ex: for $\mathcal{C}$ = rep. category of $SU(2)$, simple objects $= \{V_j\}_{j=0,1/2,1,\dots}$

$\mathrm{Hom}(V_{j_1}\otimes V_{j_2}, V_{j_3})$ - space of intertwiners $\begin{cases} 1\text{-dimensional if } (j_1 j_2 j_3) \text{ satisfy triangle inequality, with a preferred vector given by Clebsch-Gordan coeff.}\\ 0 \text{ otherwise}\end{cases}$

$\Rightarrow$ 6j-symbol is a number.

$$\begin{Bmatrix} j_1 & j_2 & j_4\\ j_3 & j_6 & j_5\end{Bmatrix} = \sum_{m_1\dots m_6} (-1)^{\sum_k (j_k - m_k)} \begin{pmatrix} j_1 & j_2 & j_4\\ -m_1 & -m_2 & m_4\end{pmatrix}\begin{pmatrix} j_1 & j_5 & j_6\\ m_1 & m_5 & -m_6\end{pmatrix}\begin{pmatrix} j_4 & j_3 & j_6\\ -m_4 & -m_3 & m_6\end{pmatrix}\begin{pmatrix} j_2 & j_3 & j_5\\ m_2 & m_3 & -m_5\end{pmatrix}$$

Quantum group

$U_q(sl_2)$ — deformation of $U(sl_2)$ in category of Hopf algebras

fix $q \neq \pm 1$

$$U_q(sl_2) = \mathbb{C}\langle E, F, K, K^{-1}\rangle \Big/ \begin{array}{l} KK^{-1} = 1 = K^{-1}K \\ q^2 EK = KE \\ FK = q^2 KF \\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \end{array}$$

co-product

$$\Delta: \begin{array}{l} E \mapsto E \otimes 1 + K \otimes E \\ F \mapsto F \otimes K^{-1} + 1 \otimes F \\ K \mapsto K \otimes K \end{array}$$

co-unit

$$\varepsilon: E \mapsto 0, \; F \mapsto 0, \; K \mapsto 1$$

antipode

$$S: \begin{array}{l} E \mapsto -K^{-1}E \\ F \mapsto -FK \\ K \mapsto K^{-1} \end{array}$$

Rem "Classical limit"

$q = e^{ih/2}, h \to 0$ $\quad K = q^H$ $\quad$ in sl_2, $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$U_q(sl_2)$ " $\xrightarrow[q \to 1]{}$ " $U(sl_2)$

Category of fin.d. representations of $U_q(sl_2)$: semi-simple, with simple objects $V_0^{(q)}, V_{\frac{1}{2}}^{(q)}, \dots, V_{\frac{r-1}{2}}^{(q)}$ $\quad \boxed{\text{Check}}$

b. q a root of unity

18.04.13 8/1 7/5

6j-symbols for classical SU(2) & Ponzano-Regge model

Ref - G. Ponzano, T. Regge, "Semiclassical limit of Racah coefficients", 1968
- J.W. Barrett "The Ponzano-Regge model", 0803.3319

Ponzano-Regge model

M - triangulated ~~compact~~ closed 3-mfd

$Z_M = \sum_{\varphi} Z_{M,\varphi}$ (*)

φ: edges $\to \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$

s.t. for $\forall$ 2-simplex j_1, j_2, j_3

$\begin{cases} j_1+j_2+j_3 \in \mathbb{Z} \\ j_1, j_2, j_3 \text{ satisfy triangle inequality} \end{cases}$

irreps of SU(2)

$$(**)\quad Z_{M,\varphi} = \left(\prod_{\text{edges}} (-1)^{2j} \cdot (2j+1)\right)\left(\prod_{\text{tetrahedra}} \underbrace{\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}}_{(-1)^{\Sigma(j_1+\dots+j_6)} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}}\right)$$

Racah-Wigner 6j-symbol for SU(2)

- Z_M is given by an infinite sum, which requires regularization
  a) cut-off for spins
  b) regularization by deforming to TV model and taking $q \to 1$.
  (c) in terms of group-variables

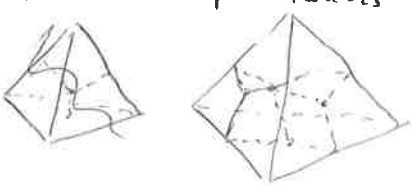
Ponzano-Regge asymptotic formula for 6j-symbols

$$(***)\quad \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \underset{\substack{j_k = \lambda \tilde{j}_k,\ k=1..6 \\ \lambda\to\infty}}{\sim} \frac{1}{\sqrt{12\pi |V|}} \cos\left(\sum_{k=1}^{6} (j_k + \tfrac{1}{2})\Theta_k + \frac{\pi}{4}\right)$$

where $|V|$ - volume of the tetrahedron with Euclidean metric, with edges of lengths $j_1+\frac{1}{2}, \dots, j_6+\frac{1}{2}$

- ~~PR model~~ (*) is a model for 3D quantum gravity; weight (**) for large spins becomes, due to (***), exp of Einstein action for 3D gravity evaluated on a piecewise-Euclidean metric, defined by lengths of edges being $j_k + \frac{1}{2}$. ??

- Dual picture: "spin foams" (or "shadows") - 2-skeletons of cell complexes dual to triangulations



2-cells of a "foam" carry spins
1-cells - admissibility conditions for 3 adjacent spins
~~0~~-cells - 6j-symbols.

- Group variables:
the weight (**) can be rewritten as

$$Z_{M,\varphi} = \prod_{\substack{\text{triangles} \\ \text{(edges of spin-foam)}}} \int_{SU(2)} dg \prod_{\substack{\text{edges} \\ \text{(2-cells of foam)}}} (2j+1)\, \mathrm{Tr}_j(h) \xrightarrow{\text{sum over spins}} \prod_{\text{triangles}} \int_{SU(2)} dg \prod_{\text{edges}} \delta(h)$$

h: holonomy around bdry of 2-cell of foam

corresponds to path integral for 3D BF theory with $G = SU(2)$, $S = \mathrm{tr} \int_M B \wedge F_A$
~ 3D gravity in 1st order formalism

- Ponzano-Regge Regularization:

$$\hat{Z}_M = N_\Lambda^{-\#\text{vertices}} \sum_{\varphi:\text{ edges}\to\{0,\frac{1}{2},1,\frac{3}{2},\dots,\Lambda\}} Z_{M,\varphi}$$

Λ ← cut off on spins

$Z_M = \lim_{\Lambda\to\infty} \hat{Z}_\Lambda$ exists for <u>some</u> triangulated manifolds

(?) $N_\Lambda = \sum_{j=0}^{\Lambda} (2j+1)$

- better regularization: via Turaev-Viro

$$Z_M = \lim_{r\to\infty} Z_M^{q=e^{2\pi i/r}}$$

Introducing knots (with group variables)
(observables)

$K \subset M$: a knot going along edges of triangulation

additional data: θ: $\{\text{edges of } K\} \to G/G$, $G = SU(2)$

G/G ← conjugacy classes

$$Z_{M,\varphi} \longmapsto Z_{M,\varphi} \cdot \prod_{\substack{\text{edges } e\\ \text{of } K}} \underbrace{\operatorname{Tr}_{\varphi(e)}(\theta(e))}_{\frac{\sin(2j+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}} \times \underbrace{\operatorname{Vol}(\theta(e))}_{\frac{\sin^2\theta/2}{\pi}} \cdot \frac{1}{2j+1}$$

$\frac{\sin^2\theta/2}{\pi}$ - volume of conjugacy class

$\frac{\sin(2j+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$ - character of $SU(2)$ in spin j representation

$$\rightsquigarrow Z_{M,K,\theta} = \prod_{\text{triangles}} \int_{SU(2)} dg \prod_{\text{edges}\notin K} \delta(h_e) \cdot \prod_{\text{edges}\in K} \delta(h_e, \theta_e)$$

Σ over spins

~~TQFT~~

Classical Chern-Simons ~~(S)~~ theory

M - 3-manifold, $\mathfrak{g} = Lie(G)$ - a Lie algebra with ad-invariant pairing $\langle , \rangle$ (usually the Killing form) $\langle a,b\rangle = Tr_{\mathfrak{g}}\, ad_a ad_b$

Rem for quantization, one usually requires that G be simple compact. (simply connected?)

$P = G \times M \to M$ — trivial principal bundle

* G-bundles on a 3-mfld are trivial

fields: $F = Conn(P) \simeq \mathfrak{g} \otimes \Omega^1(M) \ni A$

action: $S(A) = \int_M \frac{1}{2}\langle A \overset{\wedge}{,} dA\rangle + \frac{1}{6}\langle A \overset{\wedge}{,} [A, A]\rangle \underset{\text{if } \langle,\rangle = tr\, ad, ad.}{=} tr \int_M \frac{1}{2} A \wedge dA + \frac{1}{3} A\wedge A \wedge A$

Rem if M ~~closed~~ closed, $M = \partial N$, N 4-mfld, then $S(A) \underset{\text{by Stokes' thm}}{=} \frac{1}{2}\int_N \langle F_{\tilde{A}} \overset{\wedge}{,} F_{\tilde{A}}\rangle$

for $\tilde{A}$ an extension of A to a connection over N. $F_{\tilde{A}} = d\tilde{A} + \tilde{A}\wedge\tilde{A}$

Gauge symmetry: $g: M \to G$

$(A \mapsto A^g = g^{-1}Ag + g^{-1}dg) \rightsquigarrow S$

$S(A^g) = \cancel{S(A) + \frac{1}{2}\int_{\partial M} \langle A \wedge dg\, g^{-1}\rangle} = S(A) + \underbrace{\frac{1}{2}\, tr \int_{\partial M} A \wedge dg\, g^{-1}}_{= tr \int_{\partial M} A^g \wedge g^{-1}dg} - \frac{1}{6}\, tr \int_M (g^{-1}dg)^3$

Rem: $\exists\, \nu \in \mathbb{R}$ normalization constant, s.t.

$\underset{=: \frac{1}{6}\nu\beta}{\alpha} = -\frac{1}{6}\nu\, tr\, (h^{-1}dh)^3 \in \Omega^3_{closed}(G)$ is integral

For $G = SU(N)$, $tr = tr$ in fund. rep., $\nu = \frac{1}{4\pi^2}$; class $[\alpha] \in H^3(G,\mathbb{R})$ is non-trivial
$\in im(H^3(G,\mathbb{Z}))$

* for $\partial M = \emptyset$,

$S(A^g) = S(A) + \frac{1}{6}\underbrace{\int_M g^* \beta}_{\text{“}\int \langle g_*[M], [\beta]\rangle\text{”}}$

* related fact: $\pi_3(G) = \mathbb{Z}$ for G compact simple

so, for $k \in \mathbb{Z}$ "level"

$\nu k\, S(A^g) = \nu k \cdot S(A) + \underbrace{k \cdot \langle g_*[M], [\alpha]\rangle}_{\in \mathbb{Z}}$, so $\nu k \cdot S(A)$ is well-defined mod $\mathbb{Z}$ as a function on $Conn(P)/$Gauge transf.

* for $g: M \to G$ homotopic to identity, $g_*[M] = 0$ and thus $S(A^g) = S(A)$ gauge invariance

(other way to see this: infinitesimal gauge transf. $g = 1 + \varepsilon X + O(\varepsilon^2)$, $X: M \to \mathfrak{g}$
$S(A^g) = S(A) + O(\varepsilon^2)$)

Rem $g: M \to G$ not homotopic to id — "large" gauge transformations

- for $\partial M \neq \emptyset$, infinitesimal gauge transf.:

$$S(A) \longmapsto S(A) + \frac{\varepsilon}{2} \mathrm{tr} \int_{\partial M} A \wedge dX + O(\varepsilon^2)$$

- gauge symmetry is spoiled by boundary term.

Equations of motion (Euler-Lagrange eq.):

$\underbrace{F_A}_{dA + A\wedge A} = 0$ - flatness condition on A.

$$\{\text{solutions of EL eq.}\} / \text{gauge symmetry} = \{A \in \mathfrak{g} \otimes \Omega^1(M) \mid dA + A \wedge A = 0\} / A \sim g^{-1} A g + g^{-1} dg =$$

=: moduli space of flat G-connections on M $\simeq \mathrm{Hom}(\pi_1(M), G) / G$ - finite-dimensional singular variety

$MFC_G(M)$

Boundary phase space

$\underset{\text{surface}}{\Phi_\Sigma}$ = {restrictions (pull-backs) of fields (connections) on M (a 3-mfd) with $\partial M = \Sigma$ to Σ} = $\mathrm{Conn}_G(\Sigma \times G \to \Sigma) \simeq \mathfrak{g} \otimes \Omega^1(\Sigma)$

boundary term of variation of action: $\delta S(A) = -\mathrm{tr} \int \frac{1}{2}(\delta A \wedge dA + A \wedge d\delta A) + \delta A \wedge A \wedge A =$

$\underset{\text{Stokes}}{=} -\mathrm{tr} \int_M \delta A \wedge \underbrace{(dA + A \wedge A)}_{F_A \text{ - EL equation}} + \underbrace{\mathrm{tr} \frac{1}{2} \int_{\partial M} A \wedge \delta A}_{\text{boundary 1-form}}$

> Notation: d - de Rham on M, Σ
> δ - de Rham on $\mathrm{Conn}_M, \mathrm{Conn}_\Sigma$

$\rightsquigarrow$ boundary 1-form $\alpha_\Sigma = \frac{1}{2} \mathrm{tr} \int_\Sigma A \wedge \delta A \in \Omega^1(\Phi_\Sigma)$

- symplectic structure: $\omega_\Sigma = \delta \alpha_\Sigma = \frac{1}{2} \mathrm{tr} \int_\Sigma \delta A \wedge \delta A \in \Omega^2(\Phi_\Sigma)$ (weakly non-degenerate!)
- normalization

$\exp i 2\pi \underbrace{\nu k S(A)}_{S_k(A)} \in \{z \in \mathbb{C}, |z| = 1\}$ - well-defined on $\mathrm{Conn}_M / \text{gauge transf}$

$\rightsquigarrow$ $\alpha_k = 2\pi\nu k \cdot \alpha_\Sigma$
$\omega_k = 2\pi\nu k \cdot \omega_\Sigma$ - normalized symplectic structure

ω_k is $2\pi \times$ ~~integral!~~ exact

- constraint ($A \in \Phi_\Sigma$ can be extended to a sol. of EL on $\Sigma \times [0, \varepsilon]$)

$\Leftrightarrow F_A = 0$, $C_\Sigma \subset \Phi_\Sigma$, C_Σ = {flat G-connections on Σ}

- gauge transformations on Σ:

finite $A \longmapsto A^g = g^{-1} A g + g^{-1} dg$, $g : \Sigma \to G$

infinitesimal: $A \mapsto A + \varepsilon d_A X + O(\varepsilon^2)$, $X : M \to \mathfrak{g}$
i.e. X defines a vector field $v(X)$ on Φ_Σ,
$\mathrm{Map}(M, \mathfrak{g}) \xrightarrow{v} \mathfrak{X}(\Phi_\Sigma)$ - Lie algebra homomorphism

<u>Note</u> (1) • vector fields $\gamma(X)$ are Hamiltonian (w.r.t. Poisson structure $\{,\}$ generated by ω_Σ),

(*) $\gamma(X) = \{\underbrace{\mathrm{tr}\int_\Sigma X F_A}_{H_X}, \cdot\}$

<u>Rem:</u> normalized version. $\{,\}_k = \frac{1}{2\pi\nu k}\{,\}$
$H_{X,k} = 2\pi\nu k\, H_X$

(3) (*) implies that ~~gauge~~ infinitesimal gauge symmetry on Φ_Σ, viewed as a distribution on Φ_Σ, restricted to C_Σ is the characteristic distribution on C_Σ.

(2) $\{H_X\}_{X:M\to\mathfrak{g}}$ ~~span~~ generate the vanishing ideal of C_Σ

$\{H_X, H_Y\} = H_{[X,Y]}$ ← check!

$\Rightarrow C_\Sigma$ coisotropic

(4) $\mathrm{Gauge}_\Sigma = \mathrm{Map}(\Sigma, G)$

$\mathrm{Gauge}_\Sigma \curvearrowright$ Conn μ: $\Phi_\Sigma \xrightarrow{\text{curvature}} \mathfrak{g}\otimes\Omega^2(\Sigma)$

$\Phi_\Sigma = \mathrm{Conn}_G(\Sigma)$, $\mathfrak{g}\otimes\Omega^2(\Sigma) \simeq (\mathrm{gauge}_\Sigma)^*$

is the (equivariant) moment map ~~generating~~ for the (Hamiltonian) action of Gauge_Σ

(5) <u>Reduced phase space</u>:

$$\Phi_\Sigma^{red} = C_\Sigma/\mathrm{Gauge}_\Sigma = \underline{C_\Sigma} = \mu^{-1}(0)/\mathrm{Gauge}_\Sigma$$

= moduli space of flat connections on Σ

symplectic reduction — Marsden–Weinstein reduction

• Φ_Σ^{red} comes with Atiyah–Bott symplectic structure $\underline{\omega_\Sigma}$ – reduction of ω_Σ

with normalization $\underline{\omega}_{\Sigma,k} = 2\pi\nu k\,\underline{\omega}_\Sigma$, it is $2\pi\cdot$ <u>integral!</u> but <u>not</u> exact

ref: A. Weinstein "Symplectic structures on the moduli space"

Explain?

• ~~consider~~ $\alpha_{\Sigma,k}$ induces a connection ∇_k in a trivial $U(1)$-bundle of curvature $\omega_{\Sigma,k}$. hence, ~~the~~ the restriction of ∇_k to orbits of $\mathrm{Gauge}_\Sigma \curvearrowright C_\Sigma$ are <u>flat</u> (since orbits are isotropic), with ~~trivial~~ monodromy (~~…~~)

$$\begin{array}{c} U(1)\times\Phi_\Sigma \\ \downarrow \\ \Phi_\Sigma \end{array}$$

$\boxed{\cancel{\ldots}\ \alpha_\Sigma|_{C_\Sigma} = 0}$?

Hence, we can identify $U(1)$-fibers over every orbit using ∇_k

$\Rightarrow$ we get a "pre-quantum" $U(1)$-bundle $U(1)\curvearrowright L_k \to \Phi_\Sigma^{red}$ with connection $\underline{\nabla_k}$ of curvature $\underline{\omega_k}$

Important property: $L_k = (L_1)^{\otimes k}$
(follows from the construction)

<u>Note</u>: two different reasons for integrality of level k:
- well-definedness of e^{iS} under large gauge transf.
- reducibility of ~~α~~ form α_Σ on boundary.

"Evolution relation"

$$\begin{tikzcd} & EL_M \arrow[d, "\pi"] \arrow[dl, "\pi"'] \arrow[r, phantom, "\subset"] & F_M \arrow[d, "\pi"] \\ L_{M,\partial M} = \pi(EL) \arrow[r, phantom, "\subset"] & C_{\partial M} \arrow[r, phantom, "\subset"] & \Phi_{\partial M} \end{tikzcd}$$

— restriction to boundary

$L_{M,\partial M}$ – "evolution relation"

Rem: the term "relation" – from the setup where $\partial M = \overline{\partial_{in} M} \sqcup \partial_{out} M$,

$L_{M,\partial M} \subset (\Phi_{\partial_{in} M})^{op} \times \Phi_{\partial_{out} M}$ – set-theoretic relation

opposite sign of symplectic form

Theorem: $L_{M,\partial M} \subset \Phi_{\partial M}$ is Lagrangian (~~equivalently~~ thus, canonical relation)

TQFT (9/1)
21.05.13

Classical Chern-Simons theory

<u>Reminder</u> M 3-mfd (oriented, compact) with boundary $\partial M = \Sigma$ (surface), G - Lie group (connected compact)

$F_M : \mathrm{Conn}(M)$ $\circlearrowleft$ $\mathrm{Gauge}_M = \mathrm{Map}(M, G)$, $\quad S_M = \mathrm{tr}\int_M \frac{1}{2} A\wedge dA + \frac{1}{3} A\wedge A\wedge A$

for normalization $S_k = 2\pi\nu k\, S_M$ is invariant mod 2π w.r.t. Gauge_M for M closed.
$\nu = \frac{1}{4\pi^2}$, $k \in \mathbb{Z}$

<u>reduction in bulk</u>

$\underbrace{\mathrm{Flat\,Conn}(M)}_{EL_M} \hookrightarrow \underbrace{\mathrm{Conn}(M)}_{F_M}$

$\mathrm{Gauge}_M \circlearrowright EL_M$

$EL_M / \mathrm{Gauge}_M = \mathcal{M}_M = \mathrm{Hom}(\pi_1(M), G)/G$ - moduli space of flat connections.

<u>boundary phase space</u>

$\Phi_\Sigma = \mathrm{Conn}(\Sigma)$, $\quad \cdot\ \alpha_\Sigma = \frac{1}{2}\mathrm{tr}\int_\Sigma A\wedge\delta A \in \Omega^1(\Phi_\Sigma)$ - boundary term of δS_M with $\partial M = \Sigma$

- $\omega_\Sigma := \delta\alpha_\Sigma = \frac{1}{2}\mathrm{tr}\int_\Sigma \delta A\wedge\delta A \in \Omega^2(\Phi_\Sigma)$
- ω_Σ is a symplectic (in particular, non-degenerate) structure on Φ_Σ
- normalization: $\alpha_k = 2\pi\nu k\cdot\alpha_\Sigma$, $\omega_k = 2\pi\nu k\cdot\omega_\Sigma$ (then α_Σ pushes down to reduction, ω_Σ becomes integral)

- <u>Constraint</u>

($A \in \Phi_\Sigma$ can be extended to a $\underbrace{\text{sol. of EL}}_{=\text{flat connection}}$ over $\Sigma\times[0,\varepsilon]$) $\Leftrightarrow$ ($F_A = 0$)

$C_\Sigma \subset \Phi_\Sigma$
$C_\Sigma = \mathrm{FlatConn}_\Sigma$, $\Phi_\Sigma = \mathrm{Conn}_\Sigma$

- <u>Gauge transf. on Σ</u>

<u>finite</u>

$\mathrm{Gauge}_\Sigma \circlearrowright \Phi_\Sigma$

$g\cdot A = A^g = g^{-1}Ag + g^{-1}dg$

$g: \Sigma\to G$

<u>infinitesimal</u>

$A \mapsto A + \varepsilon\, d_A X + O(\varepsilon^2)$, $\quad g = 1 + \varepsilon X + O(\varepsilon^2)$

i.e. we have a map

$\underbrace{\mathrm{Map}(\Sigma, \mathfrak{g})}_{\mathrm{Lie}(\mathrm{Gauge}_\Sigma)} \xrightarrow{v} \mathfrak{X}(\Phi_\Sigma)$

$X \longmapsto \int_\Sigma d_A X \wedge \frac{\delta}{\delta A} = v(X)$

- Lie algebra homomorphism

<u>Note</u> (i) vector fields $v(X)$ are Hamiltonian,

(*) $\quad v(X) = \{\underbrace{\mathrm{tr}\int_\Sigma X F_A}_{H_X}, \cdot\}$

<u>Rem</u> normalized version:
$\{,\}_k = \frac{1}{2\pi\nu k}\{,\}$,
$H_{X,k} = 2\pi\nu k\, H_X$

(2) $\{H_X\}_{X:M\to\mathfrak{g}}$ generate the vanishing ideal of C_Σ,
$\{H_X, H_Y\} = H_{[X,Y]}$ $\Rightarrow$ C_Σ is coisotropic.

(3) (*) implies that infinitesimal gauge symmetry on Φ_Σ, viewed as a distribution on Φ_Σ, restricted to C_Σ, is the characteristic distribution on C_Σ.

(4) $\mathrm{Gauge}_\Sigma = \mathrm{Map}(\Sigma, G)$

$$\mathrm{Gauge}_\Sigma \circlearrowright \underset{\mathrm{Conn}(\Sigma)}{\overset{\mu:}{\Phi_\Sigma}} \xrightarrow{\text{curvature}} \underbrace{\mathfrak{g}\otimes\Omega^2(\Sigma)}_{\simeq (\mathrm{Gauge}_\Sigma)^*}$$

is the (equivariant) moment map for the (Hamiltonian) action of Gauge_Σ on Φ_Σ

i.e. $\gamma(X) = \{\langle X, \mu\rangle, \cdot\}$

Ref. Atiyah, Bott
- "The Yang Mills equations over Riemannian surfaces" 1982
- "The moment map and equivariant cohomology", 1984
- Alan Weinstein "The symplectic structure on moduli space"

Reduced phase space

- $\Phi_\Sigma^{red} := C_\Sigma / \mathrm{Gauge}_\Sigma \;=\; \underline{C_\Sigma} \;=\; \underline{\mu^{-1}(0)}$

  $\mathcal{M}_\Sigma$ - moduli space of flat connections on Σ; symplectic reduction; Marsden-Weinstein reduction

Rem: we use that Gauge_Σ is connected.

- Φ_Σ^{red} comes with Atiyah-Bott symplectic structure $\underline{\omega_\Sigma} \in \Omega^2(\mathcal{M}_\Sigma)$ - reduction of ω_Σ.
- With normalization
  $\underline{\omega_k} = 2\pi k\, \underline{\omega_\Sigma}$, $\underline{\omega_k}$ is 2π-integral, but not exact
  (e.g. for G compact, $\mathcal{M}_\Sigma$ is compact, so $\frac{(\underline{\omega_k})^{\frac{1}{2}\dim\mathcal{M}}}{(\frac{1}{2}\dim\mathcal{M})!}$ is a positive volume form $\Rightarrow [\underline{\omega_k}]^{\frac{1}{2}\dim\mathcal{M}} \neq 0 \Rightarrow [\underline{\omega_k}] \neq 0$)

- ω_Σ is a basic 2-form on total space of $C_\Sigma \circlearrowleft \mathrm{Gauge}_\Sigma$, $\downarrow$ $\mathcal{M}_\Sigma$, and so descends to $\underline{\omega_\Sigma}$ on base $\mathcal{M}_\Sigma$

but α_Σ is not basic (not horizontal), $\iota_{\gamma(X)}\alpha_\Sigma = -\frac{1}{2}\mathrm{tr}\int_\Sigma A\wedge d_A X = -\frac{1}{2}\mathrm{tr}\int_\Sigma (dA + 2A\wedge A)X \neq 0$ on C_Σ

but α_k has, viewed as 1-form of a connection ∇_k in trivial $U(1)$-bundle over C_Σ, ~~has trivial holonomy~~ is flat on orbits of Gauge_Σ (since curvature of ∇_k is ω_k and $\omega_k|_{\text{gauge orbit}} = 0$) and has trivial monodromy* when restricted to a gauge orbit.

So, $U(1)$-fibers over a gauge orbit can be identified by parallel transport by ∇_k, so we get a $U(1)$-bundle $L_k \downarrow \mathcal{M}_\Sigma$ ("pre-quantum line bundle") with connection $\underline{\nabla_k}$ of curvature $\underline{\omega_k}$

$\frac{9}{3}$
$\frac{10}{1}$

* Proof: consider a closed loop in a gauge orbit of a flat conn. on Σ:

$g_t : \underbrace{S^1 \times \Sigma}_{t\text{-parameter}} \to G \qquad A \in \text{Flat Conn}(\Sigma)$

loop: $t \mapsto A^{g_t}$

$p: S^1 \to \text{Flat Conn}_\Sigma$

$A \in$ Flat Conn$_\Sigma$ does not depend on t

$\text{Hol}_p(\nabla_k)$

$= e^{2\pi i k \oint_{S^1} dt \underbrace{\frac{1}{2} \int_\Sigma A^{g_t} \wedge \partial_t A^{g_t}}_{(\frac{d}{dt}) \cdot d_\Sigma (A^{g_t})}}$

$= \exp -2\pi i k \frac{1}{2} \text{tr} \int_{S^1 \times \Sigma} A^{g_t} \wedge d_t A^{g_t} = \exp -2\pi i k \frac{1}{2} \text{tr} \underbrace{\int_{S^1 \times \Sigma} (g^{-1} A g + g^{-1} d_\Sigma g) \wedge d_t (g^{-1} A g + g^{-1} d_\Sigma g)}_{(**)}$

$(**) = \text{tr} \int_{S^1 \times \Sigma} \underbrace{g^{-1} A g \wedge (-g^{-1} d_t g \cdot g^{-1} A g + g^{-1} A\, d_t g)}_{-g^{-1} A\, d_t g \cdot g^{-1} A g + g^{-1} A \wedge A\, d_t g} + \underbrace{g^{-1} A g \wedge d_t (g^{-1} d_\Sigma g) + g^{-1} d_\Sigma g \wedge d_t (g^{-1} A g)}$

$\downarrow$ ~~by cyclic property of tr~~

$-2 g^{-1} A^2 d_t g$

$\downarrow$ A is flat

$2 g^{-1} d_\Sigma A \wedge d_t g$

$\downarrow$ Stokes

$2 A \wedge d_\Sigma (d_t g \cdot g^{-1})$

$\shortparallel$

$-2A\, d_t d_\Sigma g \cdot g^{-1} + 2A\, d_t g \cdot g^{-1} d_\Sigma g \cdot g^{-1}$

$\downarrow$

$2A\, d_\Sigma g \cdot g^{-1} d_t g \cdot g^{-1} + 2A\, d_t g \cdot g^{-1} d_\Sigma g \cdot g^{-1}$

$2 g^{-1} d_\Sigma g \wedge d_t (g^{-1} A g)$

$\shortparallel$

$2 (g^{-1} d_\Sigma g \cdot g^{-1} d_t g \cdot g^{-1} A g - g^{-1} d_\Sigma g\, g^{-1} A\, d_t g)$

cancel

$+ \text{tr} \int_{S^1 \times \Sigma} \underbrace{g^{-1} d_\Sigma g \wedge d_t (g^{-1} d_\Sigma g)}_{-g^{-1} d_\Sigma g \, g^{-1} d_t g \cdot g^{-1} d_\Sigma g + g^{-1} d_\Sigma g \cdot g^{-1} d_\Sigma d_t g} = -\text{tr} \int_{S^1 \times \Sigma} (g^{-1} d_\Sigma g)^2 \wedge g^{-1} d_t g = -\frac{1}{3} \text{tr} \int_{S^1 \times \Sigma} (g^{-1} dg)^3 \qquad d = d_t + d_\Sigma$

$\downarrow$

$-(g^{-1} d_\Sigma g)^2 g^{-1} d_t g - \underbrace{d_t g \wedge d_\Sigma (g^{-1} d_\Sigma g \cdot g^{-1})}_{-d_t g\, g^{-1} d_\Sigma g\, g^{-1} d_\Sigma g\, g^{-1} + d_t g\, g^{-1} d_\Sigma g\, g^{-1} d_\Sigma g\, g^{-1}}$

$\Rightarrow \text{Hol}_p(\nabla_k) = \exp -2\pi i k \cdot \underbrace{\frac{1}{6} \text{tr} \int_{S^1 \times \Sigma} (g^{-1} dg)^3}_{\in \mathbb{Z}} = 1$

$\Leftarrow$

Note: $c_1(L_k) = [\omega_k]$ is automatically integral ($\frac{1}{2\pi}$?)

- Remark: $L_k = (L_1)^{\otimes k}$

(by construction, since we started with a trivial $U(1)$-bundle before reduction, with $\omega_k \propto k$)

- Remark: Two reasons for integrality of level k

- ~~well-definedness~~ invariance of e^{iS} under large gauge-transf
- reducibility of 1-form α_Σ on boundary

"<u>Evolution relation</u>" , $\Sigma = \partial M$

$$\begin{array}{ccccc} & EL_M & \subset & F_M & \\ & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \text{- restriction of fields} \\ L_{M,\Sigma} := \pi(EL_M) & \subset C_\Sigma & \subset & \Phi_\Sigma & \end{array}$$

$L_{M,\Sigma}$ - "evolution relation"

<u>Rem</u> term 'relation' - from setup where
$\partial M = \overline{\Sigma}_{in} \sqcup \Sigma_{out}$,
$L_{M,\partial M} \subset \Phi_{\Sigma_{in}}^{op} \times \Phi_{\Sigma_{out}}$ - set-theoretic relation

opposite sign of symplectic form

<u>Theorem</u> $L_{M,\Sigma}$ is <u>Lagrangian</u>
(thus, <u>canonical</u> relation)

In reduction, $\pi_* : \mathcal{M}_M \longrightarrow \mathcal{M}_\Sigma$ has Lagrangian image.
$\mathrm{im}\,\pi_* = \{$flat connections on Σ, extendable as <u>flat</u> into $M\}/$gauge transf.

E.g. Σ - genus h surface, M - handlebody

$\mathrm{im}\,\pi_* = \left\{ \begin{array}{l} \text{holonomies} \\ \text{around } \alpha\text{-cycles} \\ \text{are trivial} \end{array} \right\} / G$

α-cycles are contractible in M

• <u>Hamilton-Jacobi action</u>
for M closed, e^{iS_k} descends to a function on $\mathcal{M}_M$, which is <u>locally constant</u>,
since $\delta S_k \propto k \int_M \underbrace{F_A}_{0 \text{ on } EL} \delta A$

for $\partial M = \Sigma$, $e^{iS_k}\big|_{EL_M}$ satisfies
$(\delta + i\,\pi^* \alpha_k)\, e^{iS_k} = 0$ on EL_M, so $e^{iS_k}\big|_{EL_M}$ is a horizontal section of the pull-back by π of $U(1)$-bundle $\begin{array}{c} U(1)\times\Phi_\Sigma \\ \downarrow \\ \Phi_\Sigma \end{array}$, ∇_k

$\Rightarrow$ after reduction, e^{iS_k} defines a horizontal section of $\begin{array}{c}(\pi_*)^* L_k \\ \downarrow \\ \mathcal{M}_M \end{array}$

<u>Rem</u> e^{iS_k} is a nowhere-vanishing global section $\Rightarrow$ $(\pi_*)^* L_k$ is a trivial bundle, despite the fact that L_k is not.

Abelian Chern-Simons

$G = \mathbb{R}$

$$\mathrm{Gauge}_M = \Omega^0(M) \circlearrowright A \mapsto A^g = A + dg$$

$$S = \frac{1}{2}\int_M A \wedge dA$$

EL equation: $dA = 0$
gauge sym: $A \sim A + dg$

$$\begin{array}{ccccc} \underbrace{\Omega^1_{closed}(M)}_{EL_M} & \subset & F_M = \Omega^1(M) & & \\ \downarrow & & \downarrow \pi & & \\ L_{M,\Sigma} \subset \underbrace{\Omega^1_{closed}(\Sigma)}_{C_\Sigma} & \subset & \Phi_\Sigma = \Omega^1(\Sigma) & \circlearrowleft & \mathrm{Gauge}_\Sigma = \Omega^0(\Sigma) \end{array}$$

$$\alpha_\Sigma = \frac{1}{2}\int_\Sigma A \wedge \delta A$$

$L_{M,\Sigma} = \{\alpha \in \Omega^1(\Sigma) \mid \exists A \in \Omega^1(M) : dA = 0, A|_\Sigma = \alpha\}$

- <u>Rem</u> no reason to normalize S': no large gauge transformations, also α_Σ is <u>basic</u> on $\Omega^1_{closed}(\Sigma) \supset \Omega^0(\Sigma)$ $\rightsquigarrow$ no quantization of level
$$\Omega^1_{closed}(\Sigma) \to H^1(\Sigma)$$

<u>Reduction</u>

$$\begin{array}{ccc} EL_M/\mathrm{Gauge}_M & & C_\Sigma/\mathrm{Gauge}_\Sigma \\ \| & & \\ H^1(M) & \xrightarrow{\pi_*} & H^1(\Sigma) \\ [A] & \longmapsto & [A|_\Sigma] \end{array}$$

- $H^1(\Sigma)$ is symplectic, with symp. structure given by Poincaré pairing
$$([\alpha],[\beta]) = \frac{1}{2}\int_\Sigma \alpha \wedge \beta$$

- $\underline{L} := \mathrm{im}\,\pi_* \subset H^1(\Sigma)$ is <u>Lagrangian</u>
$= \{[\alpha] \in H^1(\Sigma) \mid \exists a \in \Omega^1_{closed}(M) : a|_\Sigma = \alpha\}$

(1) isotropicity:
$$\underset{[a|_\Sigma]}{([\alpha]}, \underset{[b|_\Sigma]}{[\beta])} = \int_\Sigma \alpha \wedge \beta = \int_M d(a \wedge b) = \int_M \underbrace{da}_{0} \wedge b - a \wedge \underbrace{db}_{0} = 0$$
$a, b \in \Omega^1_{closed}(M)$

(2) co-isotropicity:
$$\underline{L}^\perp = \{[\beta] \in H^1(\Sigma) \mid \underbrace{([a|_\Sigma],[\beta]) = 0}_{\| \text{ Stokes}} \ \forall a \in \Omega^1_{closed}(M)\} \underset{\text{by non-deg. of pairing between } H^1(M) \text{ and } H^2(M,\Sigma) \text{ (Lefschetz duality)}}{=} \ker c$$

$$\int_M a \wedge db = ([a], \underbrace{c[\beta]}_{\in H^2(M,\Sigma)})_{\text{Lefschetz}}$$
(some extension of β into M)

$$\cdots \to H^1(M) \xrightarrow{\pi_*} H^1(\Sigma) \xrightarrow{c} H^2(M,\Sigma) \to \cdots$$

$\Rightarrow$ $\mathrm{im}\,\pi_* = \underline{L}$ $\boxed{}$

Case $G = U(1)$

Fields, equations of motion, action – same as for $G = \mathbb{R}$

gauge symmetry is different: ~~Gau~~ $g : M \to U(1)$

Gauge_M is <u>not</u> connected, $\pi_0(\mathrm{Gauge}_M) \simeq H^1(M, \mathbb{Z}) \otimes \mathbb{Z}$: large gauge transf
(and Gauge_Σ)

$\overset{\shortparallel}{\text{moduli space of}}$ $\overset{\to}{\mathbb{Z}}$-bundles over M, since $U(1) = B\mathbb{Z}$

$= \mathrm{Hom}(\pi_1(M), \mathbb{Z}) = \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$

- reduction: $\mathcal{M}_M = \underbrace{\cancel{U(1) \otimes}}_{\text{is ab. group!}} H^1(M)$ $\mathrm{Hom}(H_1(M,\mathbb{Z}), U(1)) \cong H^1(M, U(1))$

$\downarrow \pi_*$

$\mathcal{M}_\Sigma = \cancel{U(1) \otimes} H^1(\Sigma) \cong H^1(\Sigma, U(1))$ — this is smaller than $\underline{C_\Sigma} = H^1(\Sigma)$, since Gauge_Σ is bigger than its connected comp. of 1.

$$\mathcal{M}_\Sigma = \underline{C_\Sigma} \Big/ (\text{large gauge transf})_\Sigma = \pi_0(\mathrm{Gauge}_\Sigma)$$

- LGT on M do not change S, but

$\mathcal{M}_\Sigma$ is compact $\Rightarrow$ there is an integrality condition on k for $\underline{\omega_u}$ on $\mathcal{M}_\Sigma$ to be integral

(it arises not because of monodromies of α_u on gauge orbits in C_Σ, but because of LGT_Σ)

<u>Rem</u> M ~~can~~ admits non-trivial flat $U(1)$-bundles, if $H^2(M, \mathbb{Z})$ has torsion. (e.g. $M = \mathbb{RP}^3$)

(generally, for any M: $\{\underline{\text{flat}}\ U(1)\text{-bundles over } M\} \big/ \text{iso of } U(1)\text{-bundles} \rightleftarrows$ torsion part of $H^2(M, \mathbb{Z})$)

$\rightsquigarrow$ part of $\mathcal{M}_M$ corresponds to flat ~~bund~~ <u>non-trivial</u> bundles.

"correct" space of fields: $F_M = \coprod_{\{\text{flat } U(1)\text{-bundles } \mathcal{L} \overset{\downarrow}{M}\}/\text{iso}} \mathrm{Conn}(\mathcal{L})$,

$$S_M(\underset{\substack{\uparrow \\ \text{chosen flat} \\ \text{connection in } \mathcal{L}}}{\nabla_\mathcal{L}} + \underset{\Omega^1(M)}{\overset{\pi}{a}}) := \int_M a \wedge da$$

Subtleties of $U(1)$-CS:	(1) large gauge transf. over Σ
	(1') discretization of level
	(1'') $\mathcal{M}_\Sigma \neq \underline{C_\Sigma}$
	(2) non-trivial flat $U(1)$-bundles over M.

11/2

09.05.17

12/0

Chern-Simons perturbation theory I

M - closed oriented 3-manifold $\longmapsto Z(M) = "\int_{Conn(M)} \mathcal{D}A" \, e^{iS_k(A)} \in \mathbb{C}$ (*)

$$S_k(A) = k \cdot \underbrace{2\pi\nu}_{\frac{1}{2\pi}} \operatorname{tr} \int_M \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A$$

(i) to make sense of PI (*),
we would like to employ formal stationary phase formula:

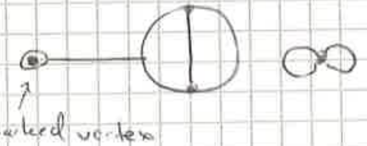
Reminder: for N a closed mfd, $\dim N = n$, $f, g \in C^\infty(N)$, f with finitely many crit. points $\{x_k^{cr}\}$, ~~$f''(x_{cr})$~~ f has non-deg Hessian f'' at x_k^{cr}, $g(x_{cr}) \neq 0$

Thm (Stationary phase formula)

$$F(\hbar) = \int_N d^n x \, g(x) \, e^{\frac{i}{\hbar} f(x)} \underset{\hbar \to 0}{\sim} \sum_{\substack{\text{crit. points} \\ x_k^{cr}}} (2\pi)^{\frac{n}{2}} e^{\frac{i}{\hbar} f(x_k^{cr})} \left(\det f''(x_k)\right)^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4} \operatorname{sign} f''(x_k)} \cdot \Big(g(x_k) +$$

$$+ \sum_{\substack{\text{graphs } \Gamma \text{ with} \\ 1 \text{ marked vertex,} \\ \text{others have valence} \geq 3}} \frac{1}{|\mathrm{Aut}\,\Gamma|} (i\hbar)^{\text{edges}} \left(\frac{i}{\hbar}\right)^{\text{vertices}} \sum_{\text{half-edges} \to (1 \ldots n)} (\partial_{\mu_1} \cdots \partial_{\mu_a} g)_{x_k} \times \prod_{\text{non-marked vertices}} (\partial_{\mu_1} \cdots \partial_{\mu_{a(v)}} f)_{x_k} \times \prod_{\text{edges } (ab)} \underbrace{(f''(x_k)^{-1})_{\mu_a \mu_b}}_{\text{"propagator"}} \Big)$$

modelled by Fresnel integrals: $\int_{-\infty}^{\infty} e^{\pm i \frac{x^2}{2}} dx = \sqrt{2\pi} \, e^{\pm \frac{\pi i}{4}}$

e.g. $\Gamma =$ 

R.h.s. is an asymptotic series in $\hbar$, i.e.

$$F(\hbar) - \sum_{j=0}^{m} F_j(\hbar) = O(\hbar^{m+1})$$

(F_j: graphs with $E - V = j$)

- Can formally extend the formula to $N = \Gamma(M, E)$ space of sections of a bundle/sheaf

(ii) critical points of integrand in (*) - flat connections - are not isolated, so we actually want to redefine

$$Z(M) = "\int_{Conn(M)/Gauge(M)} \mathcal{D}[A]" \, \underbrace{e^{iS_k(A)}}_{\text{well-defined on the quotient}}$$

Problem: not a space of sections of a bundle

(ii') Faddeev-Popov formula.

Let $G \overset{\rho}{\curvearrowright} N$ be a compact manifold with group action, $\varphi: N \to \mathfrak{g}$ s.t. $\varphi^{-1}(0) \subset N$ intersects every G-orbit transversally and only once. ("gauge-fixing" / "section of ρ")

ρ induces a family of maps $v_x: \mathfrak{g} \to T_x N$, $x \in N$ (infinitesimal action)

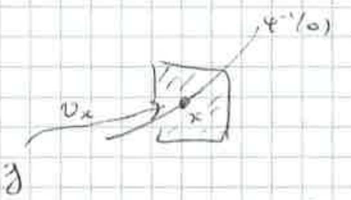
$\left(\rho: G \times N \to N, \ (D\rho)_{(1,x)}: \mathfrak{g} \times T_x N \to T_x N, \ v_x = (D\rho)_{(1,x)}(\cdot, 0) \right)$

Claim

$$\int_N \omega \, \delta(\varphi) \, \det_{\mathfrak{g}} \underbrace{(D_x \varphi \circ v_x)}_{\text{"}\partial_b \varphi^a|_{\varphi^{-1}(0)}\text{"}} \underbrace{(v_x)_* \Big(\underbrace{\mu_{\mathfrak{g}}}_{\prod_b dy^b \in \Lambda^{top} \mathfrak{g}^*}\Big)}_{\Gamma(\Lambda^{top} N^* \varphi^{-1}(0))} = \int_{\varphi^{-1}(0)} \omega|_{\varphi^{-1}(0)}$$

ω: any $(\dim N - \dim G)$-form on N

$\delta(\varphi) = \prod \delta(\varphi^a)$ - Lebesgue.



$$\mathfrak{g} \xrightarrow{v_x} T_xN \xrightarrow{D_x\varphi} \mathfrak{g}$$

Fix μ_N – a G-invariant volume form on N, & $\begin{matrix} N \\ \downarrow{\scriptstyle\pi} \\ N/G \end{matrix}$, $\underset{\text{as measure}}{\pi_* \mu_N} = \mathrm{Vol}(G)\cdot \mu_{N/G}$

For $f \in C^\infty(N)^G$, we have

$$\int_N \mu_N f = \mathrm{Vol}(G)\int_{N/G}\mu_{N/G}\, f = \mathrm{Vol}(G)\int_{\varphi^{-1}(0)}\Big(\underbrace{(\pi^*\mu_{N/G})}_{\text{as a form}}\cdot f\Big)\Big|_{\varphi^{-1}(0)} \overset{\text{by Claim}}{=}$$

$$= \mathrm{Vol}(G)\int_N \delta(\varphi)\,\underbrace{\det(D_x\varphi\circ v_x)}_{\text{Faddeev-Popov determinant}}\,\underbrace{\mu_N}_{\pi^*\mu_{N/G}\wedge(v_x)_*\mu_{\mathfrak g}}\, f$$

• We may use Fourier integral representation for δ-function: $\delta(\varphi) = (2\pi)^{-\dim\mathfrak g}\int_{\mathfrak g^*}\underbrace{\prod_a d\lambda_a}_{\mu_{\mathfrak g^*}\in\Lambda^{top}\mathfrak g} e^{i\langle\varphi,\lambda\rangle}$

and Berezin integral representation for determinant:

$$\int_{\Pi\mathfrak g\times\Pi\mathfrak g^*}\prod_a \underbrace{Dc^a D\bar c_a}_{\text{Grassmann variables}}\; e^{\sum_{a,b}\bar c_a M^a{}_b c^b} = \det M$$

Recall: (1) $\int D\theta\,(a+b\theta) = b$
(2) $\int D\theta_1\cdots D\theta_n\, \underbrace{f(\theta_1\ldots\theta_n)}_{\text{polynomial}}$ = coeff. of $\theta_n\cdots\theta_1$ in f.

Combining these, we get:

$$\int_N \mu_N f = \frac{\mathrm{Vol}(G)}{(2\pi)^{\dim\mathfrak g}}\int_{N\times\mathfrak g^*\times\Pi\mathfrak g\times\Pi\mathfrak g^*}\mu_N\,\overbrace{\prod D\lambda_a}^{\mu_{\mathfrak g^*}}\prod_a Dc^a D\bar c_a\; f\cdot e^{i\langle\varphi(x),\lambda\rangle+\langle\bar c,(D_x\varphi\circ v_x)c\rangle}$$

(iii) Large k asymptotics of $Z(M)$, M closed
assume that $\mathcal M_M$ consists of finitely many points, represented by flat connections $A^{(\alpha)}$

$$Z(M) = \text{“}\int_{\mathrm{Conn}(M)/\mathrm{Gauge}(M)}\mathcal D A\text{''}\; e^{iS_k(A)} \underset{\text{stationary phase f-la}}{\sim} \sum_{\alpha\in\mathcal M_M} e^{iS_k(A^{(\alpha)})}\cdot\delta(A^{(\alpha)})$$

$$S_k(A^{(\alpha)} + \underset{\substack{\cap \\ \mathfrak g\otimes\Omega^1(M) \\ \text{perturbation around } A^{(\alpha)}}}{a}) = S_k(A^{(\alpha)}) + 2\pi\nu k\,\mathrm{tr}\int_M\tfrac12 a\wedge d_A a + 2\pi\nu k\cdot\tfrac13\,\mathrm{tr}\int_M a\wedge a\wedge a$$

$$\delta(A^{(\alpha)}) = \int\mathcal D a\; e^{2\pi i\nu k\,\mathrm{tr}\int_M\frac12 a\wedge d_A a} \qquad (*)$$

problem: gauge symmetry $a\sim a+d_Ab$; use FP formula, with $\varphi:\Omega^1(M,\mathfrak g)\to\Omega^0(M,\mathfrak g)$,
$a\longmapsto *d_A*a$ ("Lorentz gauge")

FP: integrand of $(*)$

$$\mathrm{tr}\int_M\tfrac12 a\wedge d_A a \longmapsto \mathrm{tr}\int_M\Big(\tfrac12 a\wedge d_A a + *\lambda\wedge *d_A*a + *\bar c\wedge\Delta_A c\Big),$$

Note: we use Riem. metric for gauge-fixing

$$= \tfrac12\langle a, *d_A a\rangle + \langle\lambda, d_A*a\rangle + \langle\bar c,\Delta_A c\rangle = \langle(a+\lambda), L^-_A(a+\lambda)\rangle + \langle\bar c,\Delta_A c\rangle$$

$\langle\,,\rangle$: Hodge pairing $\mathrm{tr}\int *\alpha\wedge\beta$

$L^-_A = *d_A + d_A* : \Omega^{odd}(M,\mathfrak g)\to\Omega^{odd}(M,\mathfrak g)$ – twisted Dirac operator

Here $\lambda\in\mathfrak g^*\otimes\Omega^3$ – Lagrangian multiplier
$c,\bar c\in\Pi\mathfrak g\otimes\Omega^0$ – ghosts

ζ-regularized determinants, $\det M := e^{-\zeta_M'(0)}$

(11/1) (12/2)

$$\delta(A^{(\alpha)}) = \frac{\det(\Delta_{A^{(\alpha)}})}{\sqrt{\det L^-_{(A^\alpha)}}} = \underbrace{\frac{\det(\Delta_{A^\alpha})}{|\det L^-_{A^\alpha}|^{1/2}}}_{(*)} \cdot e^{\frac{\pi i}{2}\eta(A^{(\alpha)})}$$

Atiyah-Patodi-Singer eta-invariant,
$\eta(A^{(\alpha)}) = \frac{1}{2}\lim_{s\to 0}\sum_i \operatorname{sign}\lambda_i \cdot |\lambda_i|^{-s}$, λ_i - eigenvalues of L^-

- Norm of $\delta(A^{(\alpha)})$

Ray-Singer torsion: for E an acyclic local system on odd-dimensional manifold M,
$T(M,E) := \prod_{k=1}^{\dim M} \det(\Delta_E^{(k)})^{-\frac{(-1)^k}{2}\cdot k}$, $\Delta_E^k = \det^{\zeta\text{-reg.}}_{\Omega^k(M,E)} \Delta_E$, $\Delta_E = d_E d_E^* + d_E^* d_E$

- is independent of metric used to define Δ

(- is simple homotopy invariant of (M,E))

Rem $T(M,E) = \left[\operatorname{Sdet}_{\Omega^k_{\text{coex}}(M,E)} d^* d\right]^{1/2}$

in our case, $T(M,A^{(\alpha)}) = \Delta_{(1)}^{1/2}\Delta_{(2)}^{-1}\Delta_{(3)}^{3/2} = \Delta_{(0)}^{3/2}\Delta_{(1)}^{-1/2}$
using
$\Delta_k = \Delta_{3-k}$ (by Hodge star)
$\prod_k \Delta_{(k)}^{(-1)^k} = 1$ by Hodge theory

$$(*) = \frac{\Delta_0}{\Delta_1^{1/4}\Delta_3^{1/4}} = \Delta_0^{3/4}\Delta_1^{-1/4} = \underline{\underline{T(M,A^{(\alpha)})^{1/2}}}$$

since $(L^-)^2 = \Delta$ on $\Omega^{\text{odd}}(M,\mathfrak{g})$

- Phase of $\delta(A^{(\alpha)})$

$c_2(\text{ad})$?"

Atiyah-Patodi-Singer theorem:

(1) $\frac{\pi}{2}(\eta(A^{(\alpha)}) - \eta(0)) = h\frac{1}{2\pi}\operatorname{tr}\int_M \frac{1}{2}A^{(\alpha)}dA^{(\alpha)} + \frac{1}{3}A^{(\alpha)}\wedge A^{(\alpha)}\wedge A^{(\alpha)}$

h: dual Coxeter number of G. For $G = SU(N)$, $h = N$

(2) $\eta(0) = \dim G\cdot\eta_0$, η_0: "purely gravitational" η-invariant of $*d+d*$ on $\Omega^{\text{odd}}(M)$

combination $\underbrace{\frac{\pi}{2}\eta_0 + \frac{1}{24}S_{\text{grav }CS}(g)}$ is metric-independent. $(**)$

Here $S_{\text{grav }CS}(g) = \frac{1}{2\pi}\operatorname{tr}\int_M \frac{1}{2}\omega\wedge d\omega + \frac{1}{3}\omega\wedge\omega\wedge\omega$
- CS action evaluated on Levi-Civita connection on spin bundle of M

- $S_{\text{grav }CS}(g)$ depends on (homotopy class of) trivialization of tangent bundle TM - "framing of M"
- two trivializations of TM differ by an integer (invariantly defined) ({framings} are a torsor over $\mathbb{Z}$)
- for shift of a framing by s units,
$S_{\text{grav }CS} \longmapsto S_{\text{grav }CS} + 2\pi s$

Thus, $k\to\infty$ limit of $Z(M)$ is (incorporating the compensation $(**)$)

$$Z(M_k) \underset{k\to\infty}{\sim} e^{i\dim G\cdot(\frac{\pi}{2}\eta_0 + \frac{1}{24}S_{\text{grav }CS}(g))}\sum_\alpha e^{i(k+h)S_{CS}(A^{(\alpha)})}\cdot T(M,A^{(\alpha)})^{1/2}$$

- an invariant of framed, oriented, ~~compact closed~~ closed 3-manifold;
change of framing: $Z(M) \longmapsto Z(M)\, e^{2\pi i\frac{\dim G}{24}s}$, $s\in\mathbb{Z}$

Higher-loop corrections: Axelrod-Singer

$$Z^{pert}(\underset{\text{closed, oriented, framed}}{M}, \underset{\text{acyclic connection}}{A_0^{a}}, k) = Z^{\text{semi-classical}}(M, A_0, k) \cdot Z^{\text{higher-loop}}(M, A_0, k)$$

$$Z^{\text{higher-loop}} = \left(\exp \sum_{\ell=2}^{\infty} \left(\frac{-ik}{2\pi} \right)^{1-\ell} \sum_{\substack{\text{connected} \\ \ell\text{-loop} \\ \text{3-valent} \\ \text{graphs } \Gamma}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \pm \int_{C_{V(\Gamma)}(M)} \prod_{(ij)\in E(\Gamma)} \pi_{ij}^* \gamma \right) \cdot \underbrace{\exp\, i \left(\sum_{\ell \geq 2} \beta_\ell k^{1-\ell} \right) S_{grav CS}(g)}_{(***)}$$

where

- $C_{V(\Gamma)}(M) = \underset{\text{differential-geometric blow-up}}{\mathrm{Bl}}(M^V, \text{diagonals})$ - compact 3V-dim. manifold with corners.
- $\gamma \in \Omega^2(C_2(M), \mathfrak{g}\otimes\mathfrak{g})$ - propagator, defined by $\left(d^*_{A_0} / \Delta_{A_0} \right) f = (\pi_1)_* (\gamma \wedge \pi_2^* f)$

  where
$$\begin{array}{ccc} & C_2(M) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ M & & M \end{array}$$

- $\pi_{ij} : C_V(M) \to C_2(M)$ - picking i-th and j-th point
- $\beta_\ell \in \mathbb{R}$ - some numbers (dependent on $\mathfrak{g}$, but not on k, A_0, M)

Thm (Axelrod-Singer) (1) perturbation theory for Chern-Simons path integral (around acyclic connection, on a closed mfd) is finite in each order in k.

(2) metric-dependence can be cancelled by a universal local counter-term $(***)$, depending on framing.

Problems - extension of Z^{pert} to M with boundary (as Atiyah's TQFT)
(less ambitiously, proving consistency with surgery, e.g. $Z(M_1 \# M_2) = \frac{Z(M_1)\, Z(M_2)}{Z(S^3)}$)
- extension to non-acyclic A_0 (involves integration over $\mathcal{M}_M$)
- comparison to Witten's answers, coming from (conjectured) relation to CFT, requires to make an ad hoc shift $k \mapsto k + h$ in Z^{HP}

Space of states of Chern-Simons theory

(*) • One can extend CS by observables associated to knots/links in M:

$k_i : S^1 \to M$ knots
R_i - representation of G

$$Z(M, \{k_i, R_i\}) = \int_{\mathrm{Conn}(M)} \mathcal{D}A \; e^{ik S_{cs}(A)} \prod_i \mathrm{tr}_{R_i} \mathrm{Hol}(k_i^* A)$$

parallel transport of A along k_i)

• $\mathcal{H}_{\Sigma, \{x_i, R_i\}} = \mathrm{GeomQuant}\left(\mathcal{M}_{\Sigma, \{x_i, [h_i]\}} \; ; \; k \cdot \omega_{\Sigma, \{x_i, [h_i]\}} \; ; \; P_J \right) \ominus$

marked points on Σ; irreps of G

fixed conjugacy classes in G/G dependent on R_i and k.

complex polarization of $\mathcal{M}_{\Sigma, \{x_i, [h_i]\}}$ induced from complex structure on $\Sigma \setminus \{x_i\}$

$G \circlearrowright \mathcal{O}_i \subset \mathfrak{g}^*$ - coad. orbit corresponding to R_i

points where Wilson lines end

$$\mathcal{O}_i \xrightarrow{\exp \frac{2\pi}{k}\cdot} \mathcal{O}_i'$$

$$\begin{array}{ccc} \mathfrak{g}^* \simeq \mathfrak{g} & \xrightarrow{\exp \frac{2\pi}{k}\cdot} & G \\ \cup & & \cup \\ \mathcal{O}_i & \longmapsto & [h_i] \end{array}$$

$\ominus \; \Gamma_{hol}\left(\begin{array}{c} \mathcal{L}^{\otimes k} \\ \downarrow \\ \mathcal{M}_\Sigma \end{array} \right)$

• $\mathcal{H}_{\Sigma, \{x_i, R_i\}}$ is a finite-dimensional vector space over $\mathbb{C}$ (for G compact) with an action of affine Lie algebra $\{\hat{\mathfrak{g}}_k\}$, for every x_i and a projective action of the mapping class group of $\Sigma \setminus \{x_i\}$

• for $G = SU(2)$ and a chosen pair of pants decomposition of $\Sigma \setminus \{x_i\}$, we have a basis in $\mathcal{H}^k_{\Sigma, \{x_i, R_i\}}$ with a basis vector corresponding to every decoration of edges of Γ by numbers $\{0, 1, \ldots, k\}$

Reps of $SU(2)$ of spin $0 \le j_i \le \frac{k}{2}$

So that
- decorations of leaves are fixed to j_i
- at every vertex (λ_1, λ_2, λ_3)
  - triangle ineq. holds for $(\lambda_1, \lambda_2, \lambda_3)$, $|\lambda_1 - \lambda_2| \le \lambda_3 \le \lambda_1 + \lambda_2$
  - $\lambda_1 + \lambda_2 + \lambda_3 \in 2 \cdot \mathbb{Z}$
  - $\lambda_1 + \lambda_2 + \lambda_3 \le 2k$

  - "fusion rules" for integrable reps of $\widehat{su(2)}_k$.

• Construction of $\mathcal{H}_\Sigma$ depends on a choice of complex structure on Σ
$\rightsquigarrow$ $\mathcal{H}_\Sigma$ becomes a fiber of a vector bundle over moduli space of complex structures $\mathcal{M}_{g,n}$ with projectively flat Hitchin's connection, identifying $\mathcal{H}_{\Sigma, J}$ for different J up to a phase.

TQFT - CFT (Wess-Zumino-Witten model, associated to affine Lie algebra $\hat{\mathfrak{g}}_k$)
dictionary

space of states	space of conformal blocks - solutions to (chiral) Ward identities for $\hat{\mathfrak{g}}_k$-current
decorations of marked points - representations in which Wilson lines are taken	$\hat{\mathfrak{g}}_k$-primary fields
level k - coeff. in front of action and Atiyah-Bott symp. structure on $\mathcal{M}_\Sigma$	level of $\hat{\mathfrak{g}}_k$ - value of central element
framing dependence	projectivity of the action of mapping class group on $\mathcal{H}$.
Hitchin's connection on $\begin{array}{c}\mathcal{H}_\Sigma \to \circlearrowleft \\ \downarrow \\ M_\Sigma\end{array}$	stress-energy tensor
projective flatness of	non-primary OPE for
$k \to k+h$ shift (arising in 1-loop via η-invariant and APS theorem)	$\frac{1}{k+h}$ factor appearing in Sugawara construction for $Vir \hookrightarrow \cancel{U(\widehat{sl(2)}_k)}\ U(\hat{\mathfrak{g}}_k)$