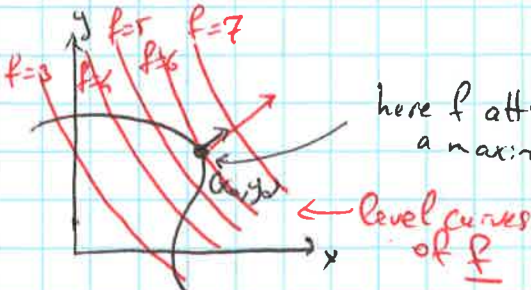
14.8 Lagrange multipliers

Problem: find extreme values of $f(x,y)$ subject to constraint $g(x,y)=k$ or: $f(x,y,z)$ subject to $g(x,y,z)=k$



$C: g(x,y)=k$

here f attains a maximum on C - level curves $g(x,y)=k$, $f(x,y)=6$ touch at (x_0,y_0)
$\Rightarrow$ have same normal line

$\Rightarrow \nabla f(x_0,y_0)$ is parallel to $\nabla g(x_0,y_0)$

i.e. $\nabla f(x_0,y_0) = \underbrace{\lambda}_{\text{a number}} \nabla g(x_0,y_0)$

a number - "Lagrange multiplier"

similarly for 3 variables:

$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)$ - level surfaces of f and g are touching

Method of Lagrange multipliers:

To find maximum & minimum values of $f(x,y,z)$ subject to constraint $g(x,y,z)=k$,

ⓐ find all values of x,y,z,λ such that
- $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ ← in components: $f_x=\lambda g_x$, $f_y=\lambda g_y$, $f_z=\lambda g_z$
- $g(x,y,z)=k$

ⓑ Evaluate f at all points (x,y,z) found in ⓐ
maximum = largest value of f
minimum = smallest ——

. Similarly for 2 variables

Ex find extreme values of $f(x,y)=x^2+2y^2$ on the circle $x^2+y^2=1$

Sol $g(x,y)=x^2+y^2=1$ - constraint

$\begin{cases}\nabla f=\lambda\nabla g\\ g(x,y)=1\end{cases}$ or $\begin{cases}f_x=\lambda g_x\\ f_y=\lambda g_y\\ g=1\end{cases}$

① $2x=\lambda\, 2x$ → $x=0$ or $\lambda=1$
② $4y=\lambda\, 2y$
③ $x^2+y^2=1$

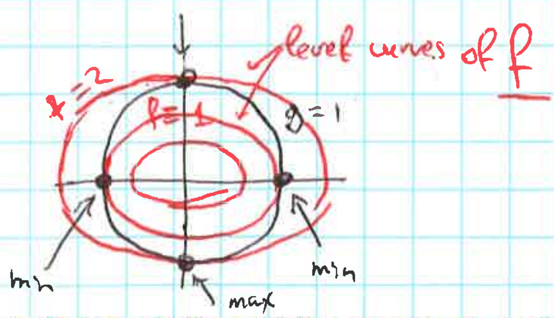
if $x=0$ then $y=\pm1$ (from ③), $\lambda=2$ from ②

if $\lambda=1$ then $y=0$ from ②, $x=\pm1$ from ③

so, four points:

(x,y)	$f(x,y)$
$(0,1)$	2
$(0,-1)$	2
$(1,0)$	1
$(-1,0)$	1

So, maximum: $\underline{2}$
minimum: $\underline{1}$



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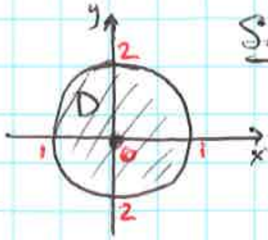
Ex: find extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2+y^2 \le 1$

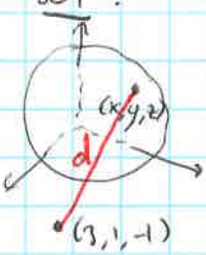


Sol ① crit. points in D: $f_x = 2x = 0$, $f_y = 4y = 0$ $\rightarrow (x,y) = (0,0)$ ← crit. point, $f(0,0) = 0$

② Extreme values of f on the boundary - from previous example:
$f(\pm 1, 0) = 1$, $f(0, \pm 1) = 2$.

So:
$$\boxed{\begin{array}{l} \text{maximum on } D: \ f(0,\pm1) = 2 \\ \text{minimum on } D: \ f(0,0) = 0 \end{array}}$$

Ex find the points on the sphere $x^2+y^2+z^2=4$ closest to and farthest from the point $P(3,1,-1)$.



Sol: $d = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$ - find extreme values / where they are attained

$d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2 = f(x,y,z)$

easier to work with than d

~~ext~~ want extreme values of f subject to $g(x,y,z) = x^2+y^2+z^2 = 4$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 4 \end{cases} \rightarrow \begin{array}{ll} 2(x-3) = \lambda 2x & ① \\ 2(y-1) = \lambda 2y & ② \\ 2(z+1) = \lambda 2z & ③ \\ x^2+y^2+z^2 = 4 & ④ \end{array}$$

$\rightarrow x - 3 = \lambda x$
$\rightarrow (1-\lambda)x = 3$

Solve ①,② ,③ ~~for~~: in terms of λ $x = \frac{3}{1-\lambda}$, $y = \frac{1}{1-\lambda}$, $z = -\frac{1}{1-\lambda}$

substitute in ④ : $x^2+y^2+z^2 = \frac{1}{(1-\lambda)^2}\underbrace{(3^2+1^2+(-1)^2)}_{11} = 4 \rightsquigarrow (1-\lambda)^2 = \frac{11}{4} \rightarrow 1-\lambda = \pm\frac{\sqrt{11}}{2}$

$\rightarrow \lambda = 1 \pm \frac{\sqrt{11}}{2}$

$\lambda = 1 + \frac{\sqrt{11}}{2} \rightsquigarrow \underbrace{\left(x = -\frac{6}{\sqrt{11}}, y = -\frac{2}{\sqrt{11}}, z = \frac{2}{\sqrt{11}}\right)}_{\text{farthest point from P on the sphere}} \leftarrow d^2 = \left(\frac{3\lambda}{1-\lambda}\right)^2 + \left(\frac{\lambda}{1-\lambda}\right)^2 + \left(-\frac{\lambda}{1-\lambda}\right)^2 = 11\left(\frac{\lambda}{1-\lambda}\right)^2 = 11\left(\frac{1+\frac{\sqrt{11}}{2}}{-\frac{\sqrt{11}}{2}}\right)^2$

$= 11\left(1+\frac{2}{\sqrt{11}}\right)^2 = (\sqrt{11}+2)^2$

$\lambda = 1 - \frac{\sqrt{11}}{2} \rightsquigarrow \underbrace{\left(x = \frac{6}{\sqrt{11}}, y = \frac{2}{\sqrt{11}}, z = -\frac{2}{\sqrt{11}}\right)}_{\text{closest point to P on the sphere}} \leftarrow d^2 = (\sqrt{11}-2)^2$