Triple integrals

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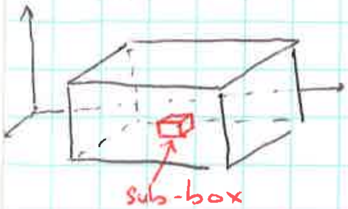


$$B = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{array} \right\}$$ rectangular box

triple integral: $$\iiint_B \underbrace{f(x,y,z)}_{\text{given (continuous) function}} \, dV = \lim_{l,m,n} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\underbrace{x_{ijk}^*, y_{ijk}^*, z_{ijk}^*}_{\text{sample point in } (i,j,k)\text{-th sub-box}}) \Delta V$$

Fubini's theorem: $$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

- 1st over x, then over y, then over z
  (y,z fixed) (z fixed)

← can be written in 5 other orders, e.g. $$\int_a^b \int_r^s \int_c^d f(x,y,z)\, dy\, dz\, dx$$

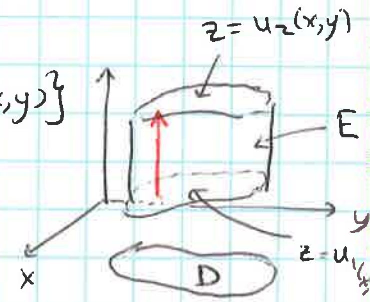
Ex: B: $0 \le x \le 1$, $-1 \le y \le 2$, $0 \le z \le 3$

$$\iiint_B xyz^2\, dV = ?$$

Sol: $$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \underbrace{\int_0^1 xyz^2\, dx}_{\left.\frac{x^2yz^2}{2}\right|_{x=0}^{x=1}}\, dy\, dz = \int_0^3 \underbrace{\int_{-1}^2 \frac{yz^2}{2}\, dy}_{\left.\frac{y^2z^2}{4}\right|_{y=-1}^{y=2}}\, dz = \int_0^3 \frac{3}{4} z^2\, dz = \left.\frac{z^3}{4}\right|_{z=0}^{z=3} = \boxed{\frac{27}{4}}$$

Triple integrals over a general region E (bounded)

Solid region E of type 1: $E = \{(x,y,z) \mid (x,y) \in \underbrace{D}_{\text{projection of E onto xy plane}},\ u_1(x,y) \le z \le u_2(x,y)\}$



$$\iiint_E f(x,y,z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dA$$

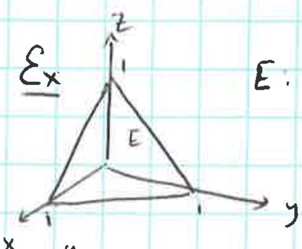
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

if D is type 1, then $E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$

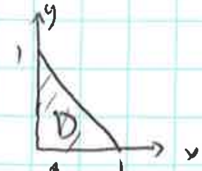
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dx\, dy$$

if D is type 2, $c \le y \le d$, $h_1(y) \le x \le h_2(y)$

Ex



E: solid tetrahedron bounded by planes $x=0$, $y=0$, $z=0$, $x+y+z=1$. find $\iiint_E z\, dV$



projection of E

$$E = \left\{ (x,y,z) \,\middle|\, \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{array} \right\}$$

Sol: $$\iiint_E z\, dV = \iint_D \left(\int_0^{1-x-y} z\, dz \right) dA = \int_0^1 \int_0^{1-x} \underbrace{\int_0^{1-x-y} z\, dz}_{\left.\frac{z^2}{2}\right|_{z=0}^{z=1-x-y}}\, dy\, dx = \frac{1}{2} \int_0^1 \underbrace{\int_0^{1-x} (1-x-y)^2\, dy}_{\left.-\frac{(1-x-y)^3}{3}\right|_{y=0}^{y=1-x}}\, dx =$$

$$= \frac{1}{6} \int_0^1 (1-x)^3\, dx = \frac{-1}{24} (1-x)^4 \Big|_0^1 = \frac{1}{24}$$

Case E: $(y,z) \in D$ ← projection onto ~~yz~~ yz plane

$u_1(y,z) \le x \le u_2(y,z)$

"type 2"

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA$$

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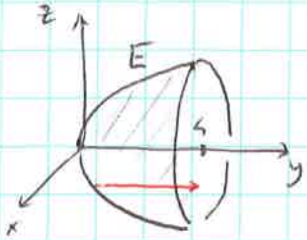
Case E: $(x,z) \in D$

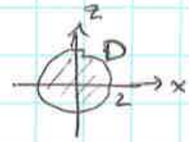
$u_1(x,z) \le y \le u_2(x,z)$ - similar

"type 3"

Ex: E bounded by $y = x^2 + z^2$, $y = 4$ find $\iiint_E \sqrt{x^2+z^2}\,dV$



Sol: View E as "type 3": $x^2 + z^2 \le 4$



~~$0 \le y \le$~~ $x^2 + z^2 \le y \le 4$

$$\iiint_E \sqrt{x^2+z^2}\,dV \cancel{= \iint_D \left(\int_0^{x^2+z^2} \sqrt{x^2+z^2}\,dy \right) dA = \iint_D (x^2+z^2)^{3/2}\,dA \underset{\text{polar coord}}{=} \int_0^{2\pi}\int_0^2 r^3\, r\,dr\,d\theta}$$

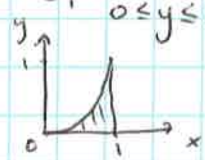
$$\cancel{= \int_0^{2\pi} \frac{r^5}{5}\Big|_{r=0}^{r=2} d\theta} = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right) dA = \iint_D (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA \underset{\text{polar}}{=} \int_0^{2\pi}\int_0^2 (4-r^2)\, r \cdot r\,dr\,d\theta$$

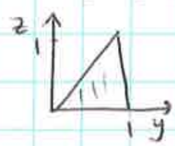
$$= \int_0^{2\pi} \left(\frac{4}{3} r^3 - \frac{r^5}{5} \right) \Big|_{r=0}^{r=2} d\theta = \frac{2}{15} \cdot 32 \cdot 2\pi = \boxed{\frac{128}{15}\pi}$$

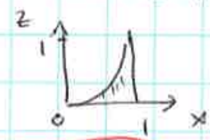
Ex: Express $I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$ as an integral in x, then in ~~y~~ z, then in y.

Sol: solid: E: $0 \le x \le 1$, $0 \le y \le x^2$, $0 \le z \le y$

projections: onto xy plane D_1: $0 \le x \le 1$, $0 \le y \le x^2$; onto yz plane D_2: $0 \le y \le 1$, $0 \le z \le y$; onto xz plane D_3: $0 \le x \le 1$, $0 \le z \le x^2$







$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

← corresponds to alternate description of E as $0 \le y \le 1$, $0 \le z \le y$, $\sqrt{y} \le x \le 1$

Applications

- $\iiint_E 1\,dV$ = Volume of the solid E
- $\iiint_E \rho(x,y,z)\,dV = m$ total mass of a solid with density function $\rho(x,y,z)$ (in kg/m^3)

center of mass: $(\bar{x}, \bar{y}, \bar{z})$: $\bar{x} = \frac{1}{m}\iiint_E x\,\rho(x,y,z)\,dV$, $\bar{y}$ = ..., $\bar{z}$ = ... similarly