14.8 Lagrange multipliers - two constraints

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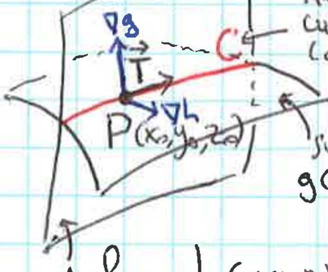
Problem: find min, max values of $f(x,y,z)$ subject to two constraints $\left\{ \begin{array}{l} g(x,y,z)=k, \\ h(x,y,z)=c \end{array}\right.$ define a curve C



if at $P(x_0,y_0,z_0)$ on C, f has a min or max, then directional derivative $D_{\vec T} f = 0$, $\vec T$ - unit tangent vector along C at P

$$D_{\vec T} f = \vec T \cdot \nabla f$$

So: $\nabla f|_P$ is orthogonal to C → lies in the plane defined by $\underbrace{\nabla g|_P, \nabla h|_P}_{\text{both orthogonal to } C}$

So: $\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$

Two Lagrange multipliers

method: look for $(x_0,y_0,z_0,\lambda,\mu)$ such that:

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x,y,z) = k \\ h(x,y,z) = c \end{cases} \qquad (*)$$

Ex: find the max value of $f(x,y,z) = x+2y+3z$ on the curve C of intersection of $\underbrace{x-y+z=1}_{g}$ (a plane) and $\underbrace{x^2+y^2}_{h}=1$ (a cylinder)

Sol: system $(*)$:

$$\begin{cases} 1 = \lambda + \mu 2x & \xrightarrow[\lambda=3]{①} 2x\mu = -2 \to x = -\frac{1}{\mu} \\ 2 = -\lambda + \mu 2y & \xrightarrow{②} 2y\mu = 5 \to y = \frac{5}{2\mu} \\ 3 = \lambda \\ x - y + z = 1 \\ x^2 + y^2 = 1 & \xrightarrow{⑤} \left(-\frac{1}{\mu}\right)^2 + \left(\frac{5}{2\mu}\right)^2 = 1 \to \frac{29}{4\mu^2} = 1 \to \mu = \pm\frac{\sqrt{29}}{2} \end{cases}$$

sol 1: $\lambda = 3$, $\mu = \frac{\sqrt{29}}{2}$, $x = -\frac{2}{\sqrt{29}}$, $y = \frac{5}{\sqrt{29}}$, $z = 1-x+y = 1+\frac{7}{\sqrt{29}}$; $f(x,y,z) = x+2y+3z = 3 + \frac{-2+2\cdot5+3\cdot7}{\sqrt{29}} = \boxed{3+\sqrt{29}}$

max value on C

sol 2: $\lambda = 3$, $\mu = -\frac{\sqrt{29}}{2}$, $x = \frac{2}{\sqrt{29}}$, $y = -\frac{5}{\sqrt{29}}$, $z = 1-x+y = 1-\frac{7}{\sqrt{29}}$; $f(x,y,z) = 3-\sqrt{29}$

(46 from textbook) problem

Ex: plane $4x-3y+8z=5$ intersects the cone $z^2=x^2+y^2$ in an ellipse C. Find highest and lowest points on C.

Sol: $f=z$ max = highest point, min = lowest point

$g = 4x-3y+8z = 5$

$h = x^2+y^2-z^2 = 0$

System of eq. on (x,y,z,λ,μ):

$$\begin{cases} 0 = 4\lambda + 2x\mu & (1)\\ 0 = -3\lambda + 2y\mu & (2)\\ 1 = 8\lambda - 2z\mu & (3)\\ 4x-3y+8z=5 & (4)\\ x^2+y^2-z^2=0 & (5)\end{cases}$$

$x = -\frac{4\lambda}{2\mu}$, $y = \frac{3\lambda}{2\mu}$, $z = \frac{8\lambda-1}{2\mu}$ or ($\mu=0$) → if $\mu=0$ then $\lambda=0$ from (1),(2) → contradicts (3)

~~or $\mu=0$~~ so, $\mu\neq 0$!

(4): $4x-3y+8z = \frac{-16\lambda-9\lambda+64\lambda-8}{2\mu} = 5 \rightsquigarrow 39\lambda-8 = 10\mu \rightsquigarrow \mu = \frac{1}{10}(39\lambda-8)$ (*)

(5): $x^2+y^2-z^2 = \frac{1}{4\mu^2}\underbrace{(16\lambda^2+9\lambda^2-64\lambda^2+16\lambda-1)}_{-39\lambda^2+16\lambda-1} = 0 \rightarrow 39\lambda^2-16\lambda+1=0$

$\rightarrow \lambda = \frac{16\pm\sqrt{16^2-4\cdot 39}}{2\cdot 39} = \frac{16\pm 10}{78} \rightarrow \lambda_1 = \frac{6}{78} = \frac{1}{13}$, $\lambda_2 = \frac{26}{78} = \frac{1}{3}$ two solutions

(A) $\lambda = \frac{1}{13} \xrightarrow{(*)} \mu = \frac{1}{10}\Big(\underbrace{\frac{39}{13}}_{3}-8\Big) = -\frac{1}{2}$

$\rightarrow \boxed{x=\frac{4}{13},\ y=-\frac{3}{13},\ z=\frac{5}{13}}$ ← lowest point on C

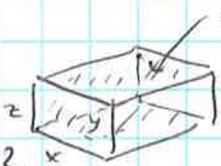
(B) $\lambda = \frac{1}{3} \xrightarrow{(*)} \mu = \frac{1}{10}\Big(\frac{39}{3}-8\Big) = \frac{1}{2} \rightsquigarrow \boxed{x=-\frac{4}{3},\ y=1,\ z=\frac{5}{3}}$ ← highest point on C

Ex: rectangular box without a lid is made out of $12\,m^2$ of cardboard. What is the maximum possible volume?

Sol: x,y,z - dimensions >0

$f(x,y,z) = xyz$ - volume

$g(x,y,z) = \underbrace{xy + \cancel{2yz}\, 2xz + 2yz}_{\text{surface area}} = 12$ ← single constant



$$\begin{cases} yz = \lambda(y+2z) \xrightarrow{\cdot\frac{1}{yz}} \lambda\left(\frac{1}{z}+\frac{2}{y}\right)=1 & (1)\\ xz = \lambda(x+2z) \xrightarrow{\cdot\frac{1}{xz}} \lambda\left(\frac{1}{z}+\frac{2}{x}\right)=1 & (2)\\ xy = \lambda(2x+2y) \xrightarrow{\cdot\frac{1}{xy}} \lambda\left(\frac{2}{y}+\frac{2}{x}\right)=1 & (3)\\ xy+2xz+2yz=12 & (4)\end{cases}$$

(1) − (2): $2\lambda\left(\frac{1}{y}-\frac{1}{x}\right)=0 \rightarrow x=y$

(1) − (3): $\lambda\left(\frac{1}{z}-\frac{2}{x}\right)=0 \rightarrow z=\frac{x}{2}$

$\rightsquigarrow$ system: $\frac{x^2}{2} = 2x\lambda$, $\frac{x^2}{2} = 2x\lambda$, $x^2 = 4x\lambda$ $\rightarrow \lambda = \frac{x}{4}$

(4) $3x^2 = 12$

$\rightsquigarrow \boxed{x=2}$, $y=2$, $z=1$

$\rightarrow$ max. volume: $2\cdot 2\cdot 1 = 4$