16.5 Curl and divergence

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$\vec{\nabla} = \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ - vector differential operator "del"

$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$ - gradient of f (scalar function)

Curl

for $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ - vector field,

$$\text{curl}\,\vec{F} = \boxed{\nabla \times \vec{F}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

easy to remember

Ex ①: $\vec{F} = \langle xz, xyz, -y^2 \rangle$ find $\text{curl}\,\vec{F}$

Sol

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz), -\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}xz\right), \frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}xz \right\rangle$$

$$= \boxed{\langle -2y - xy, x, yz \rangle}$$

• $\boxed{\text{curl}(\nabla f) = 0}$ for any f

Thus, if $\vec{F}$ is conservative, then $\text{curl}\,\vec{F} = 0$.
moreover, for $\vec{F}$ on entire $\mathbb{R}^3$, if $\text{curl}\,\vec{F} = 0$ then $\vec{F}$ is conservative

Ex ②: $\vec{F}$ from Ex ① has nonzero curl $\rightarrow$ $\vec{F}$ is not conservative!

Ex ③: $\vec{F} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$
a) show that $\vec{F}$ is conservative
b) find f such that $\vec{F} = \nabla f$

Sol: (a)
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \rangle$$
$$= \langle 0,0,0 \rangle \rightarrow \vec{F} \text{ conservative}$$

(b) $f_x = y^2z^3 \rightarrow f = xy^2z^3 + g(y,z)$

$f_y = 2xyz^3 \rightarrow 2xyz^3 + g_y(y,z) = 2xyz^3 \rightarrow g_y = 0 \rightarrow g = h(z) \rightarrow f = xy^2z^3 + h(z)$

$f_z = 3xy^2z^2 \rightarrow 3xy^2z^2 + h_z(z) = 3xy^2z^2 \rightarrow h = K \rightarrow f = xy^2z^3 + K$

• if $\text{curl}\,\vec{F} = 0$ at P, $\vec{F}$ is "irrotational" at P.

Divergence

$$\text{div}\,\underbrace{\vec{F}}_{\langle P,Q,R\rangle} = \boxed{\nabla \cdot \vec{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ - divergence of $\vec{F}(x,y,z)$

mnemonic

Ex ④ $\vec{F} = \langle xy, xyz, -y^2 \rangle$ find $\text{div}\,\vec{F}$

Sol: $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = y + xz$

• $\boxed{\text{div curl } \vec{F} = 0}$ for any $\vec{F}$!

Ex: $\vec{F}$ from Ex(5) cannot be written as curl $\vec{G}$ for some $\vec{G}$, since $\text{div } \vec{F} \neq 0$.

• if $\text{div } \vec{F} = 0$, $\vec{F}$ is "incompressible".

$\text{div } \vec{F}$ = rate of change of the mass of gas in a unit volume per unit of time.

mnemonic:

* $\boxed{\text{function} \xrightarrow{\nabla} \text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{div}} \text{function}}$

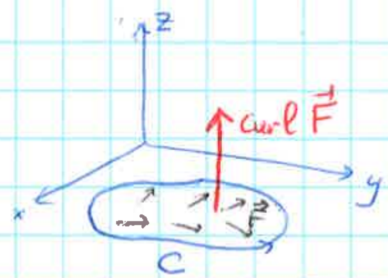
composition of any two successive arrows is zero!

* Let $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$ - indep. of z and no z-component - vector field on xy plane extended to 3D

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \rightarrow \vec{k} \cdot \text{curl } \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Green's theorem: $\oint_C \vec{F} \cdot d\vec{r} = \int_D (\text{curl } \vec{F}) \cdot \vec{k} \, dA$

curve in xy-plane; corresp region in xy-plane



* $\text{div } \nabla f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ - Laplace operator.