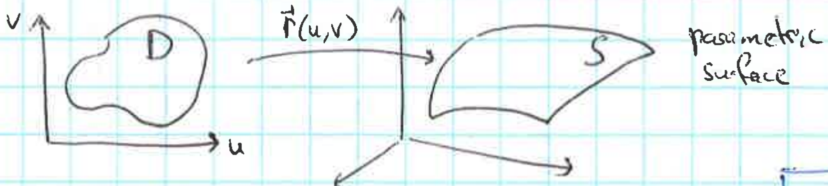
Surface integrals (16.7 cont'd)



- surface integral of a function: $\iint\limits_S f(x,y,z)\,dS = \boxed{\iint\limits_D f(\vec r(u,v))\,\underbrace{|\vec r_u \times \vec r_v|\,\overset{du\,dv}{dA}}_{dS}}$

- flux integral (surface integral of a vector field):

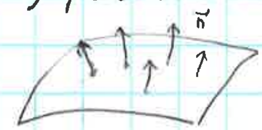
$$\iint\limits_S \vec F \cdot \vec{dS} \overset{\text{def}}{=} \iint\limits_S \vec F \cdot \vec n \, dS = \boxed{\iint\limits_D \vec F(\vec r(u,v)) \cdot (\vec r_u \times \vec r_v)\, \overset{du\,dv}{dA}}$$

for standard orientation

$\vec F(x,y,z)$ - vector field

unit normal vector field for S - defines an orientation: $\vec n = + \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|}$ ← default orientation (induced by parametrization)

or $\vec n = - \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|}$



* when orientation of S is changed, $\iint\limits_S \vec F \cdot \vec{dS}$ changes sign!

Ex: Find the flux of the vector field $\vec F(x,y,z) = \langle z, y, x \rangle$ across the unit sphere S: $x^2+y^2+z^2=1$

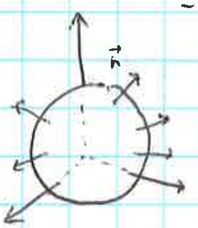
Sol: parameterize S by $\vec r(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$, $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$

$$\vec r_\varphi \times \vec r_\theta = \begin{vmatrix} \vec i & \vec j & \vec k \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix} = \langle \sin^2\varphi\cos\theta, +\sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle$$

[we computed it before]

$$= \sin\varphi \underbrace{\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle}_{\vec n \text{ - unit normal vector pointing outward}}$$

- "positive orientation"



$$\vec F(\vec r(\varphi,\theta)) = \langle \cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta \rangle$$

$$\iint\limits_S \vec F \cdot \vec{dS} = \int_0^\pi \int_0^{2\pi} (\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}} + \sin^2\varphi \overset{\cos\varphi}{\cancel{\cos^2\theta}})\,d\theta\,d\varphi$$

$$= \int_0^\pi \pi \sin^3\varphi \, d\varphi = \pi \int_0^\pi (1-\cos^2\varphi)\sin\varphi\,d\varphi \underset{-\cos\varphi = u}{=} \pi \int_{-1}^{1} (1-u^2)\,du = \pi \left| u - \frac{u^3}{3} \right|_{-1}^{1} = \frac{4\pi}{3}$$

* Flux of $\vec F = \langle P, Q, R \rangle$ through the surface S of form $z = g(x,y)$ (graph of g):

$\vec r(x,y) = \langle x, y, g(x,y) \rangle$ $\quad \vec r_x \times \vec r_y = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1 \rangle$.

So: $\iint\limits_S \vec F \cdot \vec{dS} = \iint\limits_{D \text{ in } xy\text{-plane}} \langle P,Q,R \rangle \cdot \langle -g_x, -g_y, 1 \rangle\,dA = \iint\limits_D (-P g_x - Q g_y + R)\,dA$

assuming upward orientation of S

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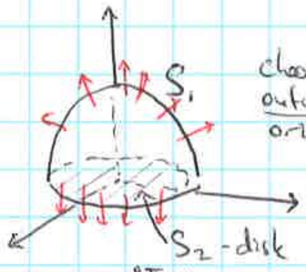
Ex: S = boundary of the solid enclosed by $z = 1-x^2-y^2$ and $z=0$,

$\vec{F} = \langle y, x, z\rangle$ Find the flux.



choose outward (positive) orientation

Sol: flux through S_1:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} \langle y, x, \underbrace{z}_{1-x^2-y^2}\rangle \cdot \langle 2x, 2y, 1\rangle\, dA = \iint_{x^2+y^2\le 1} (1-x^2-y^2+4xy)\, dA =$$

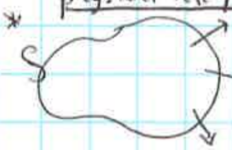
- express in terms of parameters

$$\overset{=}{\text{polar}} \int_0^{2\pi}\int_0^1 (1-r^2-\cancel{4}r^2\underbrace{2\sin\theta\cos\theta}_{2\sin 2\theta})\, r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{4}-\frac{1}{2}\sin 2\theta\right)d\theta = \frac{\pi}{2}$$

$$\iint_{S_2} \vec{F}\cdot \underbrace{d\vec{S}}_{-\vec{k}dS} = \iint_{x^2+y^2\le 1} \underbrace{\langle y, x, 0\rangle \cdot \langle 0,0,-1\rangle}_{0}\, dA = 0.$$

Thus: total flux $\iint_S \vec{F}\cdot d\vec{S} = \frac{\pi}{2}$

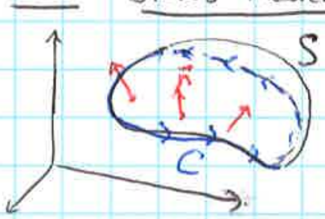
physical interpretation



* $\vec{v}$ - velocity field of the fluid of density ρ

$\iint_S \underset{\text{density}}{\rho}\, \vec{v}\cdot d\vec{S}$ - flow of the fluid (outward) through S.

16.8 Stokes' theorem



S - oriented surface (choice of $\vec{n}$), bounded by a closed curve C

C has an "induced positive orientation"

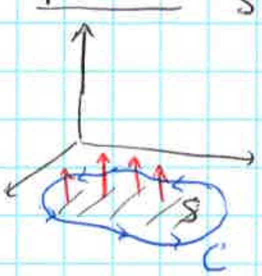
<walking along C with your head in the direction of $\vec{n}$, you should see S on your left >

Stokes' Theorem:
$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}}$$

C = "∂S" - notation

- line integral of (tangential component) of $\vec{F}$ along the boundary equals the flux of curl through the surface.

* special case: S is flat and lies in xy-plane, $\vec{n} = \vec{k}$, we get



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{\text{curl}\,\vec{F}\cdot\vec{k}}_{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}}\, dS$$

- we recovered Green's theorem!