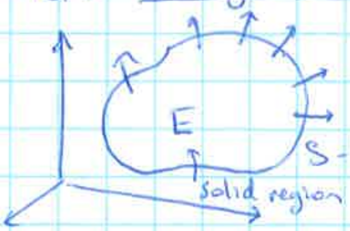
16.9 Divergence theorem

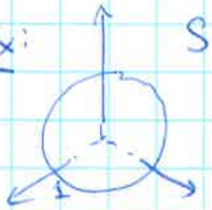


Divergence THM:

$$\boxed{\iint_S F \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV}$$

flux of F through S

Ex: S - unit sphere $x^2+y^2+z^2=1$, $\vec{F}(x,y,z) = \langle z, y, x \rangle$; find the flux $\iint_S \vec{F}\cdot d\vec{S}$

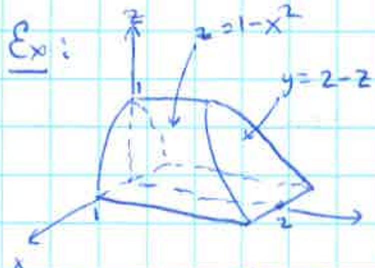


Sol: $\operatorname{div} \vec{F} = \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = 1$

$$\iint_S \vec{F}\cdot d\vec{S} \underset{\text{divergence THM}}{=} \iiint_B \operatorname{div}\vec{F}\, dV = \iiint_B 1\, dV = \text{Volume of } B = \frac{4}{3}\pi \cdot 1^3 = \frac{4\pi}{3}$$

unit ball $x^2+y^2+z^2 \le 1$

Ex:



S - surface of E bounded by: $z = 1-x^2$ - parabolic cylinder; $z=0$, $y=0$ - planes; $y+z=2$

$$\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy) \rangle$$

Find $\iint_S \vec{F}\cdot d\vec{S}$

Sol: $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2+e^{xz^2}) + \frac{\partial}{\partial z}\sin(xy) = y + 2y + 0 = 3y$

E: $\boxed{\begin{array}{l} -1 \le x \le 1 \\ 0 \le z \le 1-x^2 \\ 0 \le y \le 2-z \end{array}}$ "type 3 solid region"

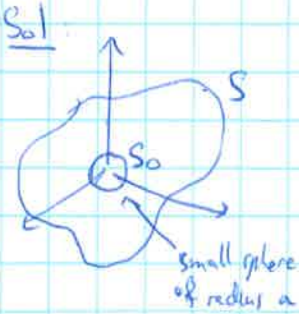
$$\rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iiint_E 3y\, dV = \int_{-1}^{1}\int_0^{1-x^2} \underbrace{\int_0^{2-z} 3y\, dy}_{\frac{3}{2}(2-z)^2}\, dz\, dx =$$

$$= \int_{-1}^{1} \underbrace{-\frac{1}{2}(2-z)^3 \Big|_{z=0}^{z=1-x^2}}_{-\frac{1}{2}\left((1+x^2)^3 - 8\right)} dx = -\frac{1}{2}\int_{-1}^{1} (x^6+3x^4+3x^2-7)\,dx = -\frac{1}{2}\cdot 2\underbrace{\left(\frac{1}{7}+\frac{3}{5}+1-7\right)}_{\frac{26}{35}-6 = -\frac{184}{35}} = \frac{184}{35}$$

Ex $\vec{E}(\vec{r}) = \frac{\varepsilon Q}{|\vec{r}|^3}\vec{r}$ - electric field of a charge Q at the origin. $\vec{r} = \langle x,y,z\rangle$

Show that ~~Find~~ the flux through any surface S enclosing the origin is $\iint_S \vec{E}\cdot d\vec{S} = 4\pi\varepsilon Q$

Sol



$$\iint_S \vec{E}\cdot d\vec{S} = \underbrace{\iint_{S_0} \vec{E}\cdot d\vec{S}}_{\frac{\varepsilon Q}{a^3}\underset{\vec{r}\cdot\vec{n}}{a}\cdot 4\pi a^2 = 4\pi\varepsilon Q} + \underbrace{\iiint_V \operatorname{div} E\, dV}_{=0}$$

check $\operatorname{div}\vec{E} =$

$$= \varepsilon Q \operatorname{div}\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$= \varepsilon Q \Big((x^2+y^2+z^2)^{-3/2} - \frac{3}{2}\cdot 2x^2(x^2+y^2+z^2)^{-5/2}$$
$$+ (x^2+y^2+z^2)^{-3/2} - 3y^2(x^2+y^2+z^2)^{-5/2}$$
$$+ (x^2+y^2+z^2)^{-3/2} - 3z^2(x^2+y^2+z^2)^{-5/2} \Big)$$

$$= \varepsilon Q\left(3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}\right) = 0$$

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Flow of a fluid of density ρ

B_a - ball centered at P_0 with small radius a

$\vec{v}$ - velocity field, $\vec{F} = \rho\vec{v}$ = rate of flow per unit area

$$\iint_{S_a} \vec{F}\cdot d\vec{S} = \iiint_{B_a} \operatorname{div}\vec{F}\, dV \approx \operatorname{div}\vec{F}(P_0)\,\underbrace{\tfrac{4}{3}\pi a^3}_{V(B_a)}$$

thus $\operatorname{div}\vec{F}(P_0) = \lim_{a\to 0} \dfrac{1}{V(B_a)} \iint_{S_a} \vec{F}\cdot d\vec{S}$

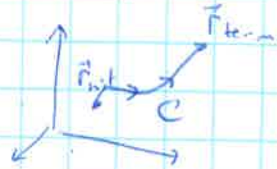
← "net rate of outward flux per unit volume at P_0"

if $\operatorname{div}\vec{F}(P_0) > 0$, net flow is <u>outward</u> (P is a "source")

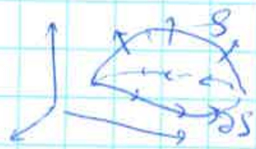
if $\operatorname{div}\vec{F}(P_0) < 0$, net flow is <u>inward</u> (P is a "sink")

<u>Summary of Stokes'-like theorems</u>:

1. fund. thm for line integrals
$$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}_{terminal}) - f(\vec{r}_{initial})$$



2. Stokes'
$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{r}$$
∂S ← bounding curve



3. divergence theorem
$$\iiint_E \operatorname{div}\vec{F}\, dV = \iint_{\partial E} \vec{F}\cdot d\vec{S}$$
∂E ← bounding surface