12.3 Dot product

Organizational: • office hours MW 4-5, HH124
• attendance sheets

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ two vectors,

$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$ - the dot product (or scalar product, or inner product)

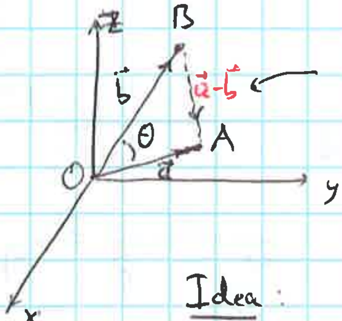
In 2D: $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$

Ex: $\langle 1, 3 \rangle \cdot \langle -2, 5 \rangle = 1(-2) + 3(5) = 13$

$(i - 2j + 5k)(\vec{j} - 2\vec{k}) = 1(0) + (-2)(1) + 5(-2) = -12$

• $\vec{a} \cdot \vec{a} = |a|^2$

• Q: does $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ make sense?



$0 \le \theta \le \pi$ - angle between vectors $\vec{a}, \vec{b}$ = angle between segments $\vec{OA}, \vec{OB}$

Theorem $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta}$

lengths (magnitudes) — angle

Idea:

law of cosines for triangle OAB: $|AB|^2 = |OA|^2 + |OB|^2 - \underbrace{2|OA|\,|OB| \cos\theta}_{2|a||b|\cos\theta}$, with $|AB|^2 = |\vec{a}-\vec{b}|^2$, $|OA|^2 = |\vec{a}|^2$, $|OB|^2 = |\vec{b}|^2$

the lhs $= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = |a|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 \Rightarrow -2\vec{a} \cdot \vec{b} = -2|a|\,|b| \cos\theta$ ☐

Ex: if $\vec{a}, \vec{b}$ have length 1, 3 and $\theta = \frac{\pi}{3}$, then $\vec{a} \cdot \vec{b} = 1 \cdot 3 \cdot \underbrace{\frac{1}{2}}_{\cos\theta} = \frac{3}{2}$.

Corollary: for $\vec{a}, \vec{b}$ nonzero vectors, $\boxed{\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}}$

Ex: $\vec{a} = \langle 1, 1, 1 \rangle$
$\vec{b} = \langle 1, 2, 3 \rangle$ find θ.

Sol: $|\vec{a}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$, $|\vec{b}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$ $\quad \vec{a} \cdot \vec{b} = 1(1) + 1(2) + 1(3) = 6$

$\Rightarrow \cos\theta = \frac{6}{\sqrt{3} \cdot \sqrt{14}} = \frac{6}{\sqrt{42}} \Rightarrow \theta = \arccos \frac{6}{\sqrt{42}} \approx 0.39 \approx 22°$

• nonzero vectors $\vec{a}, \vec{b}$ are called perpendicular (or orthogonal) if the angle is $\theta = \frac{\pi}{2}$

$\Leftrightarrow \boxed{\vec{a} \cdot \vec{b} = 0}$

Ex: $\vec{a} = \langle -1, 2, 5 \rangle$ $\quad \vec{b} = \langle 1, -2, 1 \rangle$, show that $\vec{a}, \vec{b}$ are perpendicular

Sol: $\vec{a} \cdot \vec{b} = (-1)1 + 2(-2) + 5(1) = 0$

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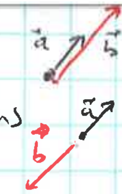
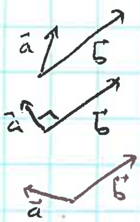
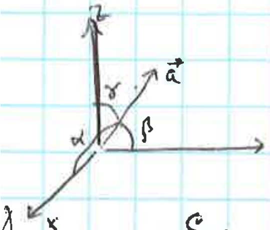
$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| \iff \theta = 0$ – $\vec{a}, \vec{b}$ point in same direction
$\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}| \iff \theta = \pi$ – $\vec{a}, \vec{b}$ point in opposite directions



$\vec{a}\cdot\vec{b} > 0$ – θ acute
$\vec{a}\cdot\vec{b} = 0$ – $\theta = \frac{\pi}{2}$
$\vec{a}\cdot\vec{b} < 0$ – θ obtuse



- for $\vec{a}$ a vector, angles α, β, γ it makes with x,y,z-axis are the <u>directional angles</u> (& their cosines – "directional cosines")



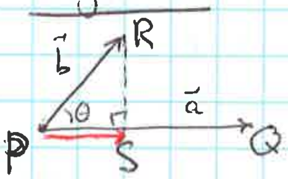
$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Thus: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\frac{1}{|\vec{a}|}\vec{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ – <u>unit vector</u> in direction of $\vec{a}$.

Ex: $\vec{a} = \langle 1,2,3 \rangle$ directional cosines: $\cos\alpha = \frac{1}{\sqrt{14}}$ ← $|\vec{a}|$

$\cos\beta = \frac{2}{\sqrt{14}}$, $\cos\gamma = \frac{3}{\sqrt{14}}$

* Projections



for $\vec{a} = \overrightarrow{PQ}$, $\vec{b} = \overrightarrow{PR}$ vectors, $\overrightarrow{PS} =: \mathrm{proj}_{\vec{a}}\vec{b}$ – vector projection of $\vec{b}$ onto $\vec{a}$.

↑ foot of ~~project~~ perpendicular dropped from R onto the line containing PQ

scalar projection (<u>component</u> of $\vec{b}$ along $\vec{a}$): $\mathrm{comp}_{\vec{a}}\vec{b} := \underbrace{|\vec{b}|\cos\theta}_{\pm|\overrightarrow{PS}|} = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}} = \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector along } \vec{a}}\cdot\vec{b}$

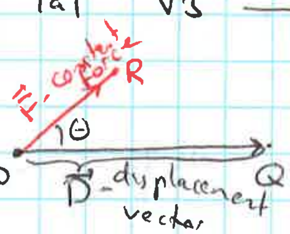
vector projection: $\mathrm{proj}_{\vec{a}}\vec{b} = \underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}_{\mathrm{comp}_{\vec{a}}\vec{b}} \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit vector along } \vec{a}} = \boxed{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a}}$

<u>Ex</u>: $\vec{b} = \langle 1,2,3 \rangle$, $\vec{a} = \langle 1,1,1 \rangle$ find $\mathrm{comp}_{\vec{a}}\vec{b}$, $\mathrm{proj}_{\vec{a}}\vec{b}$.

<u>Sol</u>: $\mathrm{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{6}{\sqrt{3}} = \underline{2\sqrt{3}}$

$\mathrm{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{6}{3}\langle 1,1,1 \rangle = \underline{\langle 2,2,2 \rangle}$

<u>Work</u> (in physics)



work $W = \underbrace{(|\vec{F}|\cos\theta)}_{\mathrm{comp}_{\vec{D}}\vec{F}}|\vec{D}| = \vec{F}\cdot\vec{D}$

<u>Ex</u>: force $\vec{F} = \langle 3,4,5 \rangle$ moves a particle from point $P(2,1,0)$ to $Q(4,6,2)$. find the work done.

<u>Sol</u>: $\vec{D} = \overrightarrow{PQ} = \langle 2,5,2 \rangle$, $W = \vec{F}\cdot\vec{D} = 3(2)+4(5)+5(2) = 36$