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## Vector functions, space curves

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vector (-valued) function : domain - set of real numbers $\longrightarrow$ range - set of vectors

component functions

$t \longmapsto \vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$
$= \langle f(t), g(t), h(t) \rangle$

indep. variable

Ex: $\vec{r}(t) = \left\langle 1+t, \dfrac{\sqrt{t}}{1-t}, \ln t \right\rangle$

domain: values $t$ for which $\vec{r}(t)$ is defined

defined for all $t \in (-\infty, \infty)$ $\qquad t \in [0,1) \cup (1,\infty)$ $\qquad t \in (0,\infty)$

so: domain is $\boxed{t \in (0,1) \cup (1,\infty)}$

### Limits

$\lim\limits_{t\to a} \vec{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t) \right\rangle$

Ex: $\lim\limits_{t\to 0} \left( \dfrac{\sin t}{t}\,\vec{i} + e^{t}\,\vec{j} + \cancel{\cos t}\ \tfrac{t^2+t}{t}\,\vec{k} \right)$

$= 1\cdot\vec{i} + 1\,\vec{j} + 1\cdot\vec{k}$

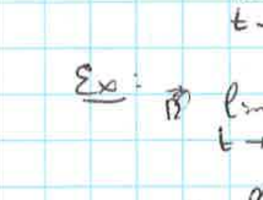
- $\vec{r}(t)$ is continuous at $a$ if $\lim\limits_{t\to a} \vec{r}(t) = \vec{r}(a)$
- $\vec{r}(t)$ is continuous at $a \iff f, g, h$ are all continuous at $a$.

### Space curves

set of points ~~(f(t), g(t), h(t))~~ $(x,y,z)$ with $\boxed{x = f(t),\ y = g(t),\ z = h(t)}$, $t \in I$ interval - "space curve" $C$

given functions

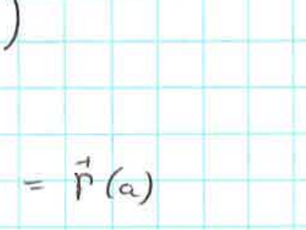
parametric equations of $C$
$t$ - parameter




vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ gives a $t$-depended ("moving") vector, whose tip traces the curve $C$.

Ex $\quad \vec{r}(t) = \langle 1, t, 2t \rangle$

param. equations $\quad x = 1$
$y = t$
$z = 2t$




describe the line through $(1,0,0)$ parallel to $\underbrace{\langle 0,1,2 \rangle}_{\vec{v}}$. Also: $\vec{r} = \underbrace{\vec{r_0}}_{\langle 1,0,0\rangle} + t\,\vec{v}$ - vector eq. of the line

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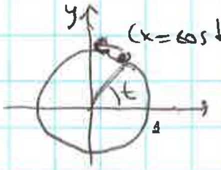
**Ex:** $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ (vector eq. of the curve) - sketch the curve

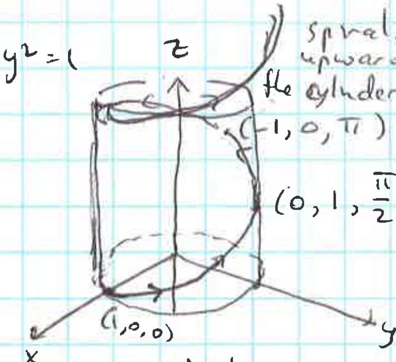
**Sol:** parametric eq: $x = \cos t$ $\quad y = \sin t$ $\quad z = t$

- $x^2+y^2 = \cos^2 t + \sin^2 t = 1 \implies \vec{r}(t)$ is on the cylinder $x^2+y^2=1$

in xy plane:




- counterclockwise rotation $\longrightarrow$ in space:




spirals upwards along the cylinder, as $t$ increases

helix

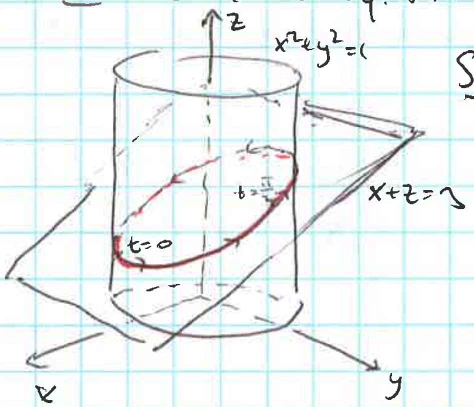
**Ex:** $P(1,1,1)$ $\quad Q(1,2,3)$ - describe the line segment $PQ$ by a vector eq.

optional

**Sol:** $\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = \langle 1, 1+t, 1+2t\rangle$, $\quad \underline{0 \le t \le 1}$

$\vec{r}_0 = \langle 1,1,1\rangle$, $\vec{r}_1 = \langle 1,2,3\rangle$, $\vec{r}_0 = \langle 1,1,1\rangle$

**Ex** find a vec. eq. for the intersection $C$ of the cylinder $x^2+y^2=1$ and plane $x+z=3$




**Sol:** ① projection of $C$ onto xy-plane is the circle $x^2+y^2=1$, $z=0$

given parametrically by $x=\cos t$, $y = \sin t$, $z=0$, $\quad 0 \le t \le 2\pi$

② from the eq. of the plane, $z = 3 - x = 3 - \cos t$

Sol: $\boxed{\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + (3-\cos t)\,\vec{k}}$, $\quad 0 \le t \le 2\pi$

- parametrization of the curve $C$

**Ex:** Find the intersection of with sphere $x^2+y^2+z^2=10$

optional

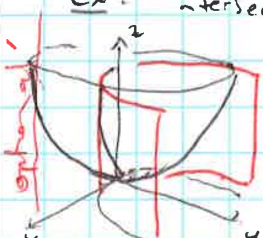
**Sol:** $\underbrace{(\cos t)^2 + (\sin t)^2 + (3-\cos t)^2}_{1 \quad + 9 - 6\cos t + \cos^2 t} = 10 \rightarrow \underbrace{\cos^2 t - 6\cos t}_{\cos t(\cos t - 6)} = 0 \rightarrow \cos t = 0$

$= 10 - 6\cos t + \cos^2 t$

$\rightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

**Ex:** intersection $C$ of $\underbrace{z = x^2+y^2}_{\text{paraboloid}}$ and $\underbrace{y = x^2}_{\text{parabolic cylinder}}$ - describe by $\vec{r}(t)$

optional



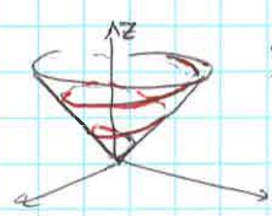

**sol:** set $t = x$ then $y = t^2$ and $z = x^2 + y^2 = 2t^2$

$\longrightarrow$ ~~Any~~ $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + 2t^2\,\vec{k}$

**Ex:** $\vec{r}(t) = (t\cos t, t\sin t, t)$, $t \ge 0$

optional

$\rightsquigarrow \underbrace{x^2+y^2}_{=t^2} = z^2$ - cone




expanding helix lying on a cone

"conical spiral"