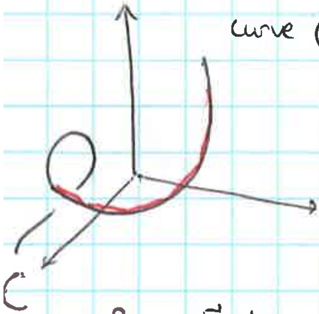
# 13.3 Arc length, [no curvature], TNB frame

09/07/2018
1



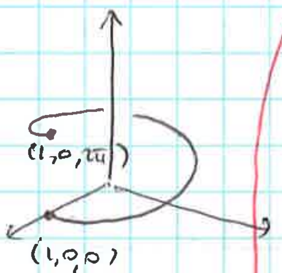

curve $C$ with vec. eq. $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ , $a \le t \le b$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt \quad \text{– length of the curve}$$

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

Ex: $\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$ (*) find the arc length between $(1,0,0)$ and $(1,0,2\pi)$

$t=0$ $\qquad$ $t=2\pi$




Sol: $\vec{r}\,'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k}$

$\rightsquigarrow |\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$

$\rightsquigarrow L = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\,\pi$

optional

- arc length function: $s(t) = \int_a^t |\vec{r}\,'(u)|\, du$ – length of the piece of the curve between $\vec{r}(a)$ and $\vec{r}(t)$

Ex: $\vec{r}(t) =$ , $a = 0$ – find the length function

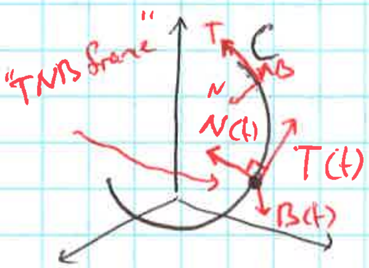
Sol: $s(t) = \int_0^t |\vec{r}\,'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$.

one can express (*) via $s$: $\quad \vec{r} = \cos \frac{s}{\sqrt{2}}\, \vec{i} + \sin \frac{s}{\sqrt{2}}\, \vec{j} + \frac{s}{\sqrt{2}}\, \vec{k}$ – parametrization of the curve via arc length

- at a point $\vec{r}(t)$ on $C$: unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

[principal] unit normal vector $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ – note that $\vec{T}(t) \cdot \vec{T}\,'(t) = 0$ !

("points in the direction "in which the curve is turning")




$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ – (unit) binormal vector

- $\{\vec{T}(t), \vec{N}(t), \vec{B}(t)\}$ – "TNB frame" – 3 orthogonal vectors

Ex: $\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\, \vec{k}$ $\quad$ find $\vec{T}, \vec{N}, \vec{B}$

Sol: $\vec{r}\,'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k}$ , $|\vec{r}\,'(t)| = \sqrt{2}$

$\Rightarrow \vec{T}(t) = \frac{1}{\sqrt{2}}(-\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k}) \longrightarrow \vec{T}\,'(t) = \frac{1}{\sqrt{2}}(-\cos t\, \vec{i} - \sin t\, \vec{j})$ , $|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}$

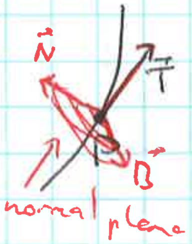
$\vec{N}(t) = \sqrt{2}\, \vec{T}\,'(t) = -\cos t\, \vec{i} - \sin t\, \vec{j}$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(\sin t\, \vec{i} - \cos t\, \vec{j} + \vec{k})$$

09/07/2018
2

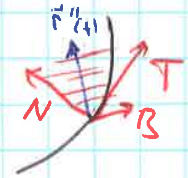
For a point $P$ on $C$, plane through $\vec N, \vec B$ — "normal plane" of $C$ at $P$

— all vectors orthogonal to $\vec T$




plane through $\vec T, \vec N$ — "osculating plane"

<for a plane curve, it is the plane containing $C$>




Ex find equations of normal & osculating planes for $\vec r(t) = \cos t\,\vec i + \sin t\,\vec j + t\,\vec k$ at $P(0, 1, \frac{\pi}{2})$ ← $t = \frac{\pi}{2}$

Sol: normal plane — passes through $P$, has normal vector $\vec T = \frac{1}{\sqrt2}\langle -1, 0, 1\rangle$ or just $\vec r'(\frac{\pi}{2}) = \langle -1, 0, 1\rangle$

So, eq: $-1(x-0) + 0(y-1) + 1(z - \frac{\pi}{2}) = 0$

or $-x + z - \frac{\pi}{2} = 0$

osculating plane: normal vector $\vec B(\frac{\pi}{2}) = \frac{1}{\sqrt2}\langle 1, 0, 1\rangle$ or just $\langle 1, 0, 1\rangle$

Eq: $1(x-0) + 0(y-1) + 1(z - \frac{\pi}{2}) = 0$ or: $x + z - \frac{\pi}{2} = 0$.

---

• Fact: $\vec r''(t)$ is in the osculating plane.

Ex: $\vec r(t) = \langle t^2, \frac23 t^3, t\rangle$ find $\vec T, \vec N, \vec B$ at $P(1, \frac23, 1)$. Find eq. of normal and osc. planes.

$t = 1$

Sol: $\vec r(1) = \langle 1, \frac23, 1\rangle$ $\quad \vec r'(t) = \langle 2t, 2t^2, 1\rangle$ $\quad \vec r''(t) = \langle 2, 4t, 0\rangle$

$\vec r'(1) = \langle 2, 2, 1\rangle$ → $\vec T = \langle \frac23, \frac23, \frac13\rangle$

$\vec r''(1) = \langle 2, 4, 0\rangle$

~~$\vec B = \vec r'(1) \times \vec r''(1)$~~

$\vec b = \vec r'(1) \times \vec r''(1)$ — normal to osc. plane ~ $\vec B = \frac{\vec b}{|\vec b|} = \frac16 \vec b = \langle -\frac23, \frac13, \frac23\rangle$

new method — first find $\vec B$, then $\vec N$.
— we avoid unpleasant computation of $\vec T'(t)$

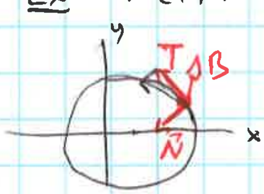
$\begin{vmatrix} \vec i & \vec j & \vec k \\ 2 & 2 & 1 \\ 2 & 4 & 0 \end{vmatrix} = \langle -4, 2, 4\rangle$

$\vec N = \vec B \times \vec T = \frac19 \begin{vmatrix} \vec i & \vec j & \vec k \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \frac19\langle -3, 6, -6\rangle = \langle -\frac13, \frac23, -\frac23\rangle$

---

Ex: $\vec r(t) = \langle 3\cos t, 3\sin t, 0\rangle$ $\quad \vec r'(t) = \langle -3\sin t, 3\cos t, 0\rangle$ → $\vec T(t) = \langle -\sin t, \cos t, 0\rangle$

$|\vec r'(t)| = 3$




$\vec T'(t) = \langle -\cos t, -\sin t, 0\rangle = \vec N(t)$

$|\vec T'(t)| = 1$

$\vec B(t) = \vec T \times \vec N = \begin{vmatrix} \vec i & \vec j & \vec k \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \vec k$