Konstantin:
Wenli:

Index theorems on manifolds with boundary

Recall: $D: \Gamma(E) \to \Gamma(E)$ elliptic partial diff. operator

$$\boxed{\mathrm{ind}(D) = t\text{-}\mathrm{ind}(D)}$$

if E is a cliff. bundle and D its Dirac operator

$$D: \Gamma(E) \xrightarrow[\nabla^E]{} \Gamma(T^*M \otimes E) \xrightarrow[cl]{} \Gamma(E)$$

connection (∇^E) cliff. multiplication (cl)

$$(*) \quad \mathrm{ind}(D_+) = \int_M \hat{A}(M)\, ch(E/S)$$

$$ch(E/S) = \mathrm{Str}^{E/S}\left(-\frac{F^{E/S}}{2\pi i}\right)$$

if M is Spin and $E = \Delta$ spinor bundle,

$$\boxed{\mathrm{ind}(D_+) = \int_M \hat{A}(M)}$$

These thms hold for closed mflds

for M closed, things don't depend on a choice of metric;
for $\partial M \neq \emptyset$, metric-dependence is very nontrivial

Q1: What happens if M has bdry?

studied by Atiyah, Patodi, Singer
"Spectral Asymmetry & Riemannian Geometry, I"

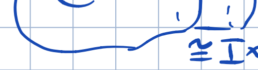
Simplest example: Gauss-Bonnet

$$\chi(X) = \frac{1}{2\pi}\int_X K + \underbrace{\frac{1}{2\pi}\int_{\partial X} \sigma}_{\text{boundary correction}}$$

X: surface; K: Gaussian curvature; σ: Geodesic curvature

Take strong assumption: metric looks like a product near the boundary, $\partial X \times [0,\varepsilon)$

- in this case the bdry correction vanishes! (for Gauss-Bonnet)



$\cong I \times S^1$

Q2 Does Theorem (*) hold for "product metric" assumption

A: NO! (*) implies

the Signature Theorem: $\mathrm{Sign}(M) = \int_M \hat{L}(M)$ (**)

think about this when M has a boundary...

$\mathrm{sign}(M)$ = signature of the bilinear form

$$\sigma : H^{2k}(M) \otimes H^{2k}(M) \longrightarrow \mathbb{R}$$

$$\sigma([\omega_1],[\omega_2]) = \int_M \omega_1 \wedge \omega_2$$

does not descend to cohom. on mflds with bdry

but $\sigma|_{H^{2k}(M,\partial M) \to H^{2k}(M)}$ descends - call its signature the signature of a mfld with bdry.

(**) is also violated for product metric assumption

reason: Stokes' theorem.

e.g. cylinder $M = [0,1] \times \underbrace{Y^3}_{\text{some 3-mfld}}$ $\mathrm{sign} = 0$

but $\int \hat{L}(M)$ depends on the metric on Y.

In general, the difference

$$\mathrm{sign}(M) - \int_M \hat{L}(M) = f(Y) \quad , Y = \partial M$$

$f(Y)$ depends only on Y and the metric on Y. (under product metric assumption)

Thought 1: $f(Y) = \int_Y \Theta$, $\Theta \in \Omega^3$ canonically constructed from metric.

$\Rightarrow f(\tilde{Y}) = p \cdot f(Y)$ - this is not true! (e.g. for $S^3 \to L_{p,q}$)

for $\tilde{Y}$ a p-fold covering of Y

$\Rightarrow f(Y)$ is a <u>global invariant</u>!

APS:

Consider the operator $B = d* \pm *d$ on Y

sign s.t. it squares to Δ

$\sigma(B) \subseteq \mathbb{R}$ can have positive and negative eigenvalues

Set $\eta(s) = \sum_{\lambda \in \sigma(B) - \{0\}} \operatorname{sign} \lambda \cdot |\lambda|^{-s}$ converges for $\operatorname{Re} s \gg 1$

-eta function of B

$\to$ can do meromorphic extension across $\mathbb{C}$.

Thm • $\eta(0)$ is finite (η invariant of Y)

• $\operatorname{sign} M = \int_M \hat{L}(M) - \eta_Y(0)$

General index theorem: M spin manifold w/ bdry Y, D its Dirac operator

$$\operatorname{ind}(D_+)_{APS} = \int_M \hat{A}(M) - \frac{n + \eta_Y(0)}{2}$$

$n = \dim \ker A$

$$D|_{\underbrace{Y \times I_u}_{\text{collar}}} = \frac{\partial}{\partial u} + \underbrace{A}_{\text{self-adj ell. operator on } Y}$$

Looking for a bdry condition for D_+ which makes the index well-defined

$\{\varphi_\lambda \mid \lambda \in \underset{\text{spectrum}}{\sigma(A)}\}$ system of eigenvectors

$P : \Gamma(E|_Y) \to \Gamma(E|_Y)$

$P\left(\sum_{\lambda \in \sigma(A)} \mu \varphi_\lambda\right) = P\left(\sum_{\lambda \geq 0} \mu_\lambda \varphi_\lambda\right)$ projector

APS bdry conditions: D_+ acts on sections whose restriction to bdry is in $\operatorname{im}(P)$ (i.e. is in the positive spectrum of A)

$(D_+)_{APS} = D : \Gamma(E, P) \to \Gamma(E)$

$\Gamma(E, P) = \{s \in \Gamma(E) \mid s|_Y \in \operatorname{im}(P)\}$

Application in physics

Consider $\quad Q(\alpha) = \int_Y \alpha \wedge d\alpha \quad$ on $\Omega^{2k-1}(Y) \qquad \dim M = 4k-1$

$\operatorname{sign} Q$ ill defined

$(\operatorname{sign} Q)_{reg}$ "=" $\eta_Y(0)$ - depends on a metric