Connections, curvatures and characteristic classes

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We follow the appendix C of [1].

1 Notations and definitions

In the following, M is a real smooth manifold and $T^*_{\mathbb{C}}M \longrightarrow M$ is the complexified cotangent bundle of M. We consider also $E \xrightarrow{\pi} M$ a complex vector bundle of rank n over M and $\{s_1, \ldots, s_n\}$ a local basis of sections of $E \xrightarrow{\pi} M$.

Definition. A connection on $E \xrightarrow{\pi} M$ is a $\mathbb{C}$-linear map

$$\nabla : \Gamma(M, E) \longrightarrow \Gamma(M, T^*_{\mathbb{C}}M \otimes E)$$

satisfying $\forall\, f \in C^\infty(M, \mathbb{C})$, $\forall\, s \in \Gamma(M, E)$,

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s) \quad \text{(Leibniz formula).}$$

Notation. We write

$$\nabla(s_i) = \sum_{j=1}^{n} \omega_{ij} \otimes s_j$$

with $\omega_{ij} \in \Gamma(M, T^*_{\mathbb{C}}M)$ and $s_j \in \Gamma(M, E)$.

Definition. We define another $\mathbb{C}$-linear map

$$\hat{\nabla} : \Gamma(M, T^*_{\mathbb{C}}M \otimes E) \longrightarrow \Gamma\left(M, \Lambda^2 T^*_{\mathbb{C}}M \otimes E\right)$$

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such that $\forall\,\theta \in \Gamma\left(M, T^*_{\mathbb{C}}M\right)$, $\forall\, s \in \Gamma\left(M, E\right)$,

$$\hat{\nabla}\left(\theta \otimes s\right) = \left(\mathrm{d}\theta\right) \otimes s - \theta \wedge \nabla\left(s\right) \quad \text{(Leibniz formula)}$$

and we define the curvature on $E \xrightarrow{\pi} M$ associated to the connection ∇ to be the $\mathbb{C}$-linear map

$$K := \hat{\nabla} \circ \nabla : \Gamma\left(M, E\right) \longrightarrow \Gamma\left(M, \Lambda^2 T^*_{\mathbb{C}}M \otimes E\right).$$

Notation. We write

$$K\left(s_i\right) = \sum_{j=1}^{n} \Omega_{ij} \otimes s_j$$

with

$$\Omega_{ij} = \mathrm{d}\omega_{ij} - \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}.$$

Remark. Everything in this section can be transposed to the real case, that is, the case where $E \xrightarrow{\pi} M$ is a real vector bundle over M, $T^*M \longrightarrow M$ is the non-complexified cotangent bundle of M,

$$\nabla : \Gamma\left(M, E\right) \longrightarrow \Gamma\left(M, T^*M \otimes E\right)$$

and

$$\hat{\nabla} : \Gamma\left(M, T^*M \otimes E\right) \longrightarrow \Gamma\left(M, \Lambda^2 T^*M \otimes E\right)$$

are $\mathbb{R}$-linear maps satisfying Leibniz formulæ.

Note that, whatever the case, M is a real smooth manifold.

Problem. We want to construct some characteristic classes of $E \xrightarrow{\pi} M$ from the curvature K.

2 Invariant polynomials, curvature and characteristic classes

Definition. An invariant polynomial on $\mathscr{M}_n\left(\mathbb{C}\right)$ is a function

$$P : \mathscr{M}_n\left(\mathbb{C}\right) \longrightarrow \mathbb{C}$$

which can be expressed as a complex polynomial in the entries of its matrix argument and satisfies

$$\forall\, X, Y \in \mathscr{M}_n\left(\mathbb{C}\right),\ P\left(XY\right) = P\left(YX\right).$$

Examples. Trace, determinant, Pfaffian...

Definition. Let $P : \mathscr{M}_n(\mathbb{C}) \longrightarrow \mathbb{C}$ be an invariant polynomial. We extend P to a map $P : \mathscr{M}_n(\mathbb{C}) \otimes \left(\bigoplus_{r=0}^{\infty} \Lambda^{2r} T^*_{\mathbb{C}} M \right) \longrightarrow \bigoplus_{r=0}^{\infty} \Lambda^{2r} T^*_{\mathbb{C}} M$

Proposition. Set $\Omega = (\Omega_{ij})_{1 \leq i,j \leq n}$. Then $P(\Omega)$ does not depend on $\{s_i\}_{1 \leq i \leq n}$ and defines local differential forms that piece together to yield a global differential form $P(K)$.

Lemma. For any invariant polynomial P, $P(K)$ is closed, i.e. $\mathrm{d}P(K) = 0$.

Remark. This is also true for P a formal power series since the powers higher than $\dim(M)/2$ are zero.

Consequence. $P(K)$ is a de Rham cocycle.

Proposition. The cohomology class of $P(K)$ is independent from the choice of the connection ∇.

Consequence. P defines a characteristic cohomology class in $H^*(M, \mathbb{C})$ depending only on the isomorphism class of $E \xrightarrow{\pi} M$.

Remark. Everything in this section remains true in the real case briefly introduced at the end of the first section, provided we consider invariant polynomials in $\mathscr{M}_n(\mathbb{R})$ instead of $\mathscr{M}_n(\mathbb{C})$.

3 Metric in the real case

In this section, we consider the real case mentioned at the end of each of the previous sections. Furthermore, assume $E \xrightarrow{\pi} M$ is provided with the Euclidean metric defining an inner product $\langle \cdot, \cdot \rangle$ such that

$$\forall \theta \in \Gamma(M, T^*M), \forall s, s' \in \Gamma(M, E), \langle \theta \otimes s, s' \rangle = \langle s, \theta \otimes s' \rangle = \langle s, s' \rangle \theta$$

Definition. A connection ∇ on E is metric, or compatible with the metric, if

$$\forall s, s' \in \Gamma(M, E), \mathrm{d} \langle s, s' \rangle = \langle \nabla s, s' \rangle + \langle s, \nabla s' \rangle$$

Notation. Let $U \subset M$ sufficiently small so that $E \xrightarrow{\pi} M$ is locally trivial above U. We denote $E|_U = U \times \mathbb{R}^n$.

Proposition. Let $\{s_1, \ldots, s_n\}$ be an orthonormal basis for $E|_U$ i.e. so that $\langle s_i, s_j \rangle = \delta_{ij}$. Then, ∇ on $E|_U$ is compatible with the metric if and only if $\omega = (\omega_{ij})_{1 \leq i,j \leq n}$ is skew symmetric.

We consider now the case $E = T^*M$

Definition. A connection ∇ on $T^*M \xrightarrow{\pi} M$ is symmetric (or torsion free) if

$$\Gamma(M, T^*M) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes T^*M) \xrightarrow{\wedge} \Gamma\left(M, \Lambda^2 T^*M\right)$$

is equal to the exterior derivative d.

Proposition. The cotangent bundle $T^*M \xrightarrow{\pi} M$ of a Riemann manifold possesses one and only one symmetric connection compatible with its metric.

Definition. This unique connection is called the Levi-Civita connection.

4 Gauss-Bonnet theorem and generalizations

Let Σ be an oriented 2-dimensional Riemannian manifold and $\{\theta_1, \theta_2\}$ a local basis of the space of 1-forms. Then

$$\omega = \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix} \qquad \text{and} \qquad \Omega = \begin{pmatrix} 0 & \Omega_{12} \\ -\Omega_{12} & 0 \end{pmatrix}$$

with $\Omega_{12} = -\,\mathrm{d}\omega_{12}$.

Definition. Write $\Omega_{12} = \mathcal{K}\,\mathrm{d}A = -\mathcal{K}\theta_1 \wedge \theta_2$. Hence, Ω_{12} is a globally defined object but not $\mathcal{K}$. Ω_{12} and $\mathcal{K}$ are respectively called the Gauss-Bonnet 2-form and the Gaussian curvature of the oriented 2-dimensional Riemannian manifold Σ.

Remark. This unusual choice for the sign of $\mathrm{d}A$ is justified in [1].

Theorem (Gauss-Bonnet)**.** For any closed oriented 2-dimensional Riemannian manifold Σ, for any oriented vector bundle $E \xrightarrow{\pi} \Sigma$ of rank 2 with Euclidean metric,

$$\iint_\Sigma \Omega_{12} = \iint_\Sigma \mathcal{K}\,\mathrm{d}A = 2\pi e_E\left([\Sigma]\right)$$

where $e_E([\Sigma])$ is the Euler class of $E \xrightarrow{\pi} \Sigma$ evaluated on the fundamental class of Σ.

In particular, if we consider the vector bundle $E = T\Sigma \xrightarrow{\pi} \Sigma$ then $e_{T\Sigma}([\Sigma])$ is simply the Euler characteristic $\chi(\Sigma)$ of Σ.

Remark. This result can be generalized to the case where Σ has a boundary.

Proposition. Let $E \xrightarrow{\pi} M$ be an oriented real vector bundle of rank $2n$ with Euclidean metric. The local $2n$-forms $\mathrm{Pf}(\Omega) \in \Gamma\left(M, \Lambda^{2n} T^*M|_U\right)$ can be pieced together to obtain a global $2n$-form $\mathrm{Pf}(\mathcal{K}) \in \Gamma\left(M, \Lambda^{2n} T^*M\right)$.

Proposition. $\mathrm{Pf}(\mathcal{K})$ is closed, thus it represents a characteristic cohomology class in $H^{2n}(M, \mathbb{R})$.

These remarks on the Pfaffian make it possible to generalize Gauss-Bonnet theorem as:

Theorem (Chern-Gauss-Bonnet)**.** For any closed oriented $2n$-dimensional Riemannian manifold M, for any oriented vector bundle $E \xrightarrow{\pi} M$ of rank $2n$ with Euclidean metric and any compatible connection,

$$\int_M \mathrm{Pf}(\Omega) = \int_M \mathrm{Pf}(\mathcal{K})\, \mathrm{d}V = (2\pi)^n\, e_E([M])\,.$$

where $\mathrm{d}V = (-1)^n\, \theta_1 \wedge \ldots \wedge \theta_n$ and $e_E([M])$ is the Euler class of $E \xrightarrow{\pi} M$ evaluated on the fundamental class of M.

In particular, if we consider the vector bundle $E = TM \xrightarrow{\pi} M$ then $e_{TM}([M])$ is simply the Euler characteristic $\chi(M)$ of M.

Remark. This result can be generalized to the case where M has a boundary.

Example. As an illustration of the Chern-Gauss-Bonnet theorem, consider S^{2n} with the Levi-Civita connection and $\{\theta_1, \ldots, \theta_{2n}\}$ an orthonormal basis for the sections of $T^*M|_U$. Then

$$-\Omega_{ij} = \theta_i \wedge \theta_j$$

and

$$\mathrm{Pf}(\Omega) = (-1)^n\, \mathrm{Pf}(\theta_i \wedge \theta_j) = (-1)^n\, (1 \cdot 3 \cdot \ldots (2n-1))\, \theta_1 \wedge \ldots \wedge \theta_{2n}$$

Furthermore, on the one hand,

$$\int_{S^{2n}} \mathrm{Pf}(\Omega) = \int_{S^{2n}} (-1)^n\, \mathrm{Pf}(\theta_i \wedge \theta_j) = (1 \cdot 3 \cdot \ldots (2n-1))\, \mathrm{Vol}\left(S^{2n}\right)$$

and on the other hand,

$$\int_{S^{2n}} \mathrm{Pf}(\Omega) = (2\pi)^n e_{TM}\left(\left[S^{2n}\right]\right) = (2\pi)^n \chi\left(S^{2n}\right) = 2\,(2\pi)^n$$

As a consequence, we recover the well-known formula for the volume of S^{2n}:

$$\mathrm{Vol}\left(S^{2n}\right) = \frac{2\,(2\pi)^n}{1 \cdot 3 \cdot \ldots \cdot (2n-1)}$$

Remark. The Euler class cannot be computed from the curvature of an arbitrary connection. We need the connection to be compatible with the metric. To illustrate the importance of this assumption, an example of a vector bundle with a flat connection but with non-zero Euler class is presented in [1]. If this connection was compatible with any metric, then, by Gauss-Bonnet theorem, the Euler class would be zero since it is computed from the Pfaffian of the curvature of this connection.

The Gauss-Bonnet theorem and the Chern-Gauss-Bonnet theorem exhibit a link between topology and differential geometry. Those results can be understood in the more general framework of Chern-Weil theory which relies on the following construction.

Definition. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of $\mathbb{C}$-valued polynomials on $\mathfrak{g}$. Let $\mathbb{C}[\mathfrak{g}]^G$ be the subalgebra of fixed points in $\mathbb{C}[\mathfrak{g}]$ under the adjoint action of G, that is, it consists of all polynomials P such that for any $g \in G$ and $X \in \mathfrak{g}$, $P(\mathrm{Ad}_g X) = P(X)$.

Given a principal G-bundle $\mathcal{P} \longrightarrow M$, there is an associated homomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow H^*(M, \mathbb{C})$$

called the Chern–Weil homomorphism, where $H^*(M, \mathbb{C})$ is the complex valued de Rham cohomology. This homomorphism is obtained by taking invariant polynomials in the curvature of any connection on the given bundle.

An important result from the Chern Weil theory is

Theorem (Chern-Weil)**.** If G is either compact or semi-simple, then the cohomology ring of the classifying space for G-bundles BG is isomorphic to the algebra of invariant polynomials:

$$H^*(BG, \mathbb{C}) \cong \mathbb{C}[\mathfrak{g}]^G .$$

Remark. This works also for $\mathbb{R}$ instead of $\mathbb{C}$.

As a conclusion and a foretaste for the coming lectures, let us mention that Atiyah-Singer index theorem, that establishes the equivalence between some quantities coming from topology on the one hand and differential geometry on the other hand, can also be regarded as a generalization of the Gauss-Bonnet theorem.

References

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