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Convexity theorem

Motivation: Delzant theorem

$$\left\{ \begin{array}{l} \text{symplectic compact} \\ \text{connected } M^{2n} \\ \text{equipped with Hamiltonian} \\ \underbrace{\mathbb{T}^n}_{n\text{-torus}}\text{- action} \end{array} \right\} \Big/ \begin{array}{l}\mathbb{T}^n\text{-equivariant} \\ \text{symplectomorphisms}\end{array} \xleftrightarrow{1-1}$$

not immediately!

$$\longleftrightarrow \left\{ \begin{array}{c} \text{convex polyhedra} \subset \mathbb{R}^n \\ \text{of special kind} \end{array} \right\} \Big/ \text{translations}$$

$$M \longmapsto \mu(M)$$

image of the moment map.
– convex polyhedron!

In fact, there is a more general statement:

Thm (Atiyah - Guillemin - Sternberg)

(M, ω) symp. mfd, connected, compact

$\mathbb{T}^m \longrightarrow \mathrm{Symp}(M, \omega)$ - Hamiltonian torus action with moment map $\mu: M \to \mathbb{R}^m$ (m does not have to be $\frac{1}{2} \dim M$)

Then: 1) $\underbrace{\mathrm{Fix}_{\mathbb{T}}(M)}_{\text{fixed points}} = \bigcup_{j=1}^{N} C_j$ ← symplectic submanifolds

2) $\mu|_{C_j} \equiv \eta_j \in \mathbb{R}^m$ const maps

!! 3) $\mu(M)$ - convex hull of η_j for all $1 \le j \le N$

4) $\mu^{-1}(\eta)$ is connected for every regular value $\eta \in \mu(M)$.

Ex: $M = \mathbb{CP}^n$, $\mathbb{T}^n$ acts on M by

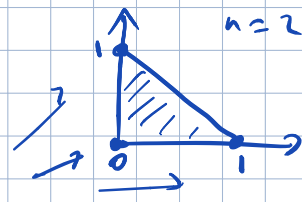
$$(\theta_1, \dots, \theta_n) \circ [z_0 : z_1 : \dots : z_n] = [z_0 : e^{2\pi i \theta_1} z_1 : \dots : e^{2\pi i \theta_n} z_n]$$

$\mu: \mathbb{CP}^n \longrightarrow \mathbb{R}^n$

$$[z_0 : \cdots : z_n] \longmapsto \left(\frac{|z_1|^2}{\|z\|^2}, \ldots, \frac{|z_n|^2}{\|z\|^2} \right), \quad \|z\|^2 = |z_0|^2 + \ldots + |z_n|^2$$

$$\mu(M) = \{ x \in \mathbb{R}^n, \ x_i \geq 0, \ \sum_{i=1}^{n} x_i \leq 1 \}$$

$n = 2$



$\mathrm{Fix}_{\mathbb{T}}(\mathbb{CP}^2) = \{[1:0:0], [0:1:0], [0:0:1]\}$

Proof of (4) ⇒ (3):

by induction in m.

for $m = 1$: $\mu : M \to \mathbb{R}$ trivial

assume for m, show for $m+1$

$\mu : M \to \mathbb{R}^{m+1}$ $\quad x_0, x_1 \in \mu(M) \subset \mathbb{R}^{m+1}$

it is enough to show that $\exists \ p : \mathbb{R}^{m+1} \to \mathbb{R}^m$, $p(x_0) = p(x_1)$, and $p^{-1} p(x_0) \cap \mu(M)$ is connected

We can show this only for p with integer coeffs

p given by a matrix $A \in \mathbb{Z}^{m \times (m+1)}$

then we have a sub-torus

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "A^T"] \arrow[d] & \mathbb{R}^{m+1} \arrow[d] \\ \mathbb{T}^m \arrow[r, hook, "A^T"] & \mathbb{T}^{m+1} \end{tikzcd}.$$

for a smaller torus we know this by induction hypothesis

any point in space of projectors can be approximated by integer projectors.

Talk 2 | Atiyah - Guillemin - Sternberg convexity theorem

Thm $T^m \xrightarrow{\psi} \mathrm{Symp}(M,\omega)$ M - compact, connected, symplectic

assume ψ is Hamiltonian, $\mu: M \to \mathbb{R}^m$

Then: 1) $\mathrm{Fix}_{T^m}(M) = \coprod_{j=1}^{N} C_j$, C_j - symp. submanifold

2) $\mu(C_j) = \eta_j \in \mathbb{R}^m$

3) $\mu(M)$ - convex hull of $\{\eta_j\}_{j=1}^N$

4) $\mu^{-1}(\eta) \subset M$ is connected for every regular value $\eta \in \mathbb{R}^m$.

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* already proved (4) $\Rightarrow$ (3)

let's prove (4).   Show for $m=1$ <then can prove by induction for $m>1$>

- Hamiltonian S^1-action on (M,ω)

$H: M \to \mathbb{R} \rightsquigarrow dH \to \underset{\text{ham. v.f.}}{X_H}$ with periodic flow.   (want to prove that it implies $H^{-1}(\{c\})$ connected)

Ex $\mathbb{R}/\mathbb{Z} = S^1 \circlearrowright \mathbb{C}$, $\frac{i}{2}\, dz \wedge d\bar{z}$

$\theta \cdot z = e^{2\pi i \lambda \theta} z$ parameter of the action, $\in \mathbb{Z}$

$H: \mathbb{C} \longrightarrow \mathbb{R}$

$z \longmapsto \lambda |z|^2$

$S^1 \circlearrowright \mathbb{C}^n$ $\theta \cdot (z_1, \dots, z_n) = (e^{2\pi i \lambda_1 \theta} z_1, \dots, e^{2\pi i \lambda_n \theta} z_n)$

$H: \mathbb{C}^n \longrightarrow \mathbb{R}$

$(z_1, \dots, z_n) \longmapsto \lambda_1 |z_1|^2 + \dots + \lambda_n |z_n|^2$ $(*)$

$S^1 \circlearrowright (\underset{\text{v.sp.}}{V}, \underset{\text{Hermitian form}}{h})$, $S^1 \to \underset{\text{unitary transf.}}{U(V,h)}$ $z \longmapsto \exp(2\pi i \underset{\text{skew-Hermitian matrix}}{A} t) \cdot z$

$H: \mathbb{C}^n \to \mathbb{R}$

$z \longmapsto h(z, Az)$

- preimage of a point is connected
- in $(*)$ in real coords, even # of pluses,

even # of minuses

Morse theory

def 1) $f: M \to \mathbb{R}$; $p \in M$ is a critical point of f if $d_pf: T_pM \to \mathbb{R}$ is zero.

2) $f: M \to \mathbb{R}$ is a Morse-Bott function if

a) $\underset{\{\text{crit. points}\}}{\mathrm{Crit}(f)} \subset M$ is a submanifold

b) for $p \in \mathrm{Crit}(f)$, $T_p(\mathrm{Crit}(f)) = \ker \underbrace{H_p(f)}_{\text{Hessian of } f \text{ at } p}$

Rem a Morse fun: $\mathrm{Crit}(f) = \coprod$ isolated points, Hessians are non-deg.

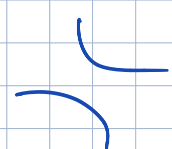
Prop: $f: M^n \to \mathbb{R}$ Morse-Bott fun., $p \in \mathrm{Crit}(f)$

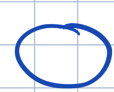
$\exists$ loc. coords $(x_1, \dots, x_n)$ near $p \in M$ s.t.

$$f(x_1, \dots, x_n) = 0 \cdot x_1^2 + \dots + 0 \cdot x_k^2 + x_{k+1}^2 + \dots + x_{k+m}^2 - x_{k+m+1}^2 - \dots - x_n^2 + c$$

- $k = \dim \mathrm{Crit}(f)$
- $m = \mathrm{coindex}_p(f)$
- $n-k-m = \mathrm{index}_p(f)$

Lemma: If $f: M \to \mathbb{R}$ is a Morse-Bott function without crit. pts of indices 1 and coindices 1, then $f^{-1}(y)$ is connected for every regular value $y \in \mathbb{R}$.

non Ex: $f: \mathbb{R}^2 \to \mathbb{R}$
$f(x,y) = x^2 - y^2$ $\quad f^{-1}(y \neq 0) =$  disconnected!

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$
$f(x,y) = x^2 + y^2$ $\quad f^{-1}(y > 0)$  connected!

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back to the proof of (4) of AGS

def V - $\mathbb{R}$-vect. space, $J: V \to V$ is a cx structure if $J^2 = -\mathrm{Id}$

Def (V, ω) symp vsp.

Then I is called compatible with ω if

(1) $\omega(Ix, Iy) = \omega(x,y)$

(2) $\omega(x, Iy)$ is inner product

<u>Lemma</u> 1) Any symp. v.sp. has a compatible complex structure

2) G - compact connected Lie group, $G \to Symp(V,\omega)$,

then $\exists$ compatible G-invariant complex structure on (V,ω)

< constructed by averaging >

$S^1 \longrightarrow Symp(M,\omega)$ $\quad$ H - Hamiltonian

1) $Fix_{S^1}(M) = Crit(H)$ $\quad$ choose S^1-invariant cx structure on M $\rightsquigarrow$ S^1-invariant Riemannian structure

2) $p \in M$, $\underset{\text{Jacobian}}{S^1 \circlearrowright T_pM}$ $\quad$ $exp: T_pM \longrightarrow M$ - S^1-equiv. map

$\underset{\text{open}}{U} \subset, \quad U \xrightarrow[\sim]{\text{homeo}} V, \quad V \subset$ open