Physics

m: mass
v: velocity
p: momentum
c: speed of light
$\hbar$: reduced Plank constant
Ψ: wave function.

$$E = \frac{1}{2} m v^2 = \frac{1}{2m} p^2$$

classical mechanics

quantum

$$i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \Delta \Psi$$

(Schödinger equation)

relativistic

$$E = \sqrt{m^2c^4 + p^2c^2}$$

(energy-momentum relation)

???

$$i\hbar \partial_t \Psi = \sqrt{\Delta}\, \Psi \,?$$

(no mass)

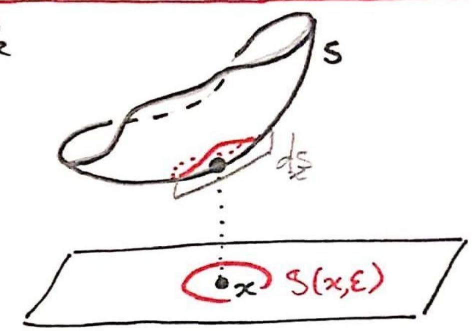
(potential Dirac equation)

Laplace and Dirac operators

$\mathbb{R}^m \times \mathbb{R}^k$

$\downarrow$

$\mathbb{R}^m$



$E = \mathbb{R}^m \times \mathbb{R}^k$
$\downarrow$
$M = \mathbb{R}^m$

vector bundle, $s : M \to E$ section.

$\Delta s(x)$ measures how far $s(x)$ is from the mean of s over a small ball:

$$\Delta s(x) := \lim_{\varepsilon \to 0} \frac{2m}{\varepsilon^2} \left(s(x) - \fint_{S(x,\varepsilon)} s(y)\, dy \right)$$

$\uparrow$ normalised integral such that $\fint_{S(x,\varepsilon)} 1\, dy = 1$

On a manifold, we need to know | how to compare vectors $\updownarrow$ connection ∇^E on E

and | what is the sphere $S(x,\varepsilon)$. $\updownarrow$ metric g on M.

normalised Riemannian integral

$$\Delta s(x) := \lim_{\varepsilon \to 0} \frac{2m}{\varepsilon^2} \left(s(x) - \fint_{S(x,\varepsilon)} /\!/_{y\to x} (s(y))\, dy \right)$$

parallel transport from y to x along the geodesic

Rk: It is the connection Laplacian, and it coincides with the Bochner Laplacian for ∇^E preserving some metric on E.

Laplace and Dirac operators

Δ is a second order operator $\mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$.

In $\begin{cases} \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \\ \text{euclidean metric} \\ \text{flat connection} \end{cases}$, a section s is given by $(s^1, \dots, s^k) : \mathbb{R}^m \to \mathbb{R}^k$.

Then the Laplacian is $\boxed{\Delta s : x \mapsto \left(-\sum_i \partial_i^2 s^1(x), \dots, -\sum_i \partial_i^2 s^k(x)\right)}$

In $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$, one can choose normal coordinates $(x^1, \dots, x^m)$ on M

(i.e. $(x^1(\gamma(t)), \dots, x^m(\gamma(t))) = tu$ for some $\begin{cases} u \in \mathbb{R}^m \text{ vector} \\ \gamma : [0, \varepsilon) \to M \text{ curve,} \end{cases}$

then γ is a geodesic) and a basis $(\varepsilon_1, \dots, \varepsilon_k)$ of E such that ε_α is parallel above the geodesics (i.e. $\nabla^E_{\dot\gamma(t)} \varepsilon_\alpha = 0$ for γ as above). Then

the inverse of g_{ij}

$$\boxed{\Delta s(x_0) = -g^{ij}(x_0)\, \partial_{ij} s^\alpha(x_0)\, \varepsilon_\alpha(x_0)}$$

the point such that $(x^1(x_0), \dots, x^k(x_0)) = (0, \dots, 0)$

Einstein convention: $\sum_{i\, j\, \alpha}$

Rk: One can show that for $s \in \mathcal{C}^\infty(E)$ fixed, the map

vector fields $\hookrightarrow X, Y \longmapsto (\nabla_X \nabla_Y s)(x) - (\nabla_{\nabla_X Y} s)(x)$

depends only on $X(x), Y(x)$, and bilinearly so. Hence

$$\text{“}\nabla^2 s\text{”} : TM \otimes TM \longrightarrow E$$
$$u \otimes v \longmapsto (\nabla_X \nabla_Y s)(x) - (\nabla_{\nabla_X Y} s)(x)$$

$u, v \in T_x M$; X, Y any vector fields such that $X(x) = u$, $Y(x) = v$

is well-defined, and its trace is Δs. Δ

A Dirac operator $D : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ is any* 1^{st} order operator such that $D^2 = \Delta + (1^{st}\text{ order operator})$.

*linear

From Dirac operators to Clifford algebras

Rk (isolated): Over $\begin{smallmatrix} M \times \mathbb{R} \\ \downarrow \\ M \end{smallmatrix}$, there can be no Dirac operator if $\dim M > 1$.

Indeed, sections of this bundle are functions $f : M \to \mathbb{R}$, so 1^{st} order operators must be of the form $D : f \mapsto \underbrace{Xf}_{= df(X)} + pf$.

X vector field ; $p : M \to \mathbb{R}$ function.

Then $D^2 f = X^2 f + $ (terms involving only f and df).

If X is not zero at some point, $X = \partial_1$ locally, so

$$D^2(x \mapsto x_2^2) = \partial_1^2(x \mapsto x_2^2) + \underset{\text{lower order}}{0} = 0$$

$$\Delta(x \mapsto x_2^2) = (-\partial_1^2 - \dots - \partial_m^2)(x \mapsto x_2^2) = -2$$

and $D^2 \neq \Delta +$ (lower order). ← for instance a Dirac candidate Δ

Every 1^{st} degree operator* $D : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ must factor through

$$\begin{tikzcd}
\mathcal{C}^\infty(E) \arrow[rr, "D"] \arrow[d, "\nabla^E"'] & & \mathcal{C}^\infty(E) \\
\mathcal{C}^\infty(T^*M \otimes E) \arrow[urr] & \lambda \otimes s \arrow[r, mapsto] & x \mapsto \rho_x(\lambda(x))(s(x))
\end{tikzcd}$$

(Intuitively, $\nabla^E s$ is the matrix of partial derivatives of s).

for some $\rho : x \longmapsto (\rho_x : T_x^* M \to \mathrm{End}(E_x))$.

This is just a reformulation of the fact that a linear first order operator is a linear combination of partial derivatives.

*linear, "without constant terms". Up to constant terms would be more accurate.

From Dirac to Clifford^{2}

Let D be a Dirac operator, and let us try to find the associated ρ. (We assume D "has no constant term," which does not change the main point since a change in the 0^{th} order of D gives a change in the 1^{st} order of D^2 which is of little importance. If one wants to consider another term, one would have to consider $D = D_\rho + m$.)

D_ρ: Constructed from ρ: $\rho : M \to T^*M \times_M \mathrm{End}(E)$ as before

m: collection of matrices: $m_x : s(x) \in E \to m_x(s(x)) \in E$

For any basis $(\varepsilon_1, \cdots, \varepsilon_k)$ of E (ε_α actually depends on the base point: $\varepsilon_\alpha(x) \in E_x$), we can write the connection ∇^E in coordinates:

$$\nabla^E \varepsilon_\alpha = \omega_\alpha^\beta \otimes \varepsilon_\beta \quad \text{for some } \omega_\alpha^\beta \in \mathcal{C}^\infty(T^*M).$$

Going further, we can take a local chart for M, say $(x^1, \dots, x^n)$ so that ω_α^β decomposes as $(\omega_\alpha^\beta)_i\, dx^i$:

$$\nabla^E \varepsilon_\alpha = (\omega_\alpha^\beta)_i \, dx^i \otimes \varepsilon_\beta.$$

Using the Leibniz rule, we get for a general section $s = s^\alpha \varepsilon_\alpha$:

$$\nabla^E(s^\alpha \varepsilon_\alpha) = ds^\alpha \otimes \varepsilon_\alpha + s^\alpha \omega_\alpha^\beta \otimes \varepsilon_\beta$$
$$= \left(\partial_i s^\alpha + s^\beta (\omega_\beta^\alpha)_i\right) dx^i \otimes \varepsilon_\beta$$

so $D(s^\alpha \varepsilon_\alpha) = \left(\underline{\partial_i s^\alpha} + s^\beta(\omega_\beta^\alpha)_i\right) \rho(dx^i)(\varepsilon_\alpha)$.

We want to understand the 2^{nd} order term of D^2, so we are only interested in the term where we differentiate $\underline{\partial_i s^\alpha}$ again:

$$D^2(s^\alpha \varepsilon_\alpha) = \partial_{ij} s^\alpha \rho(dx^i)\rho(dx^j)(\varepsilon_\alpha)$$
+ terms involving only lower derivatives of s.

Now we can show* that $\Delta(s^\alpha \varepsilon_\alpha) = -g^{ij}\partial_{ij} s^\alpha \varepsilon_\alpha$ + terms of lower order.

* For instance, use the Rk in "Laplace and Dirac".

So we must have

$$\sigma_{ij}\, \rho(dx^i)\rho(dx^j) = -g^{ij}\sigma_{ij} \underset{\uparrow \text{ identity of } E}{\mathbb{1}}$$

for all σ_{ij} symmetric. This is in fact equivalent to

$$\frac{1}{2}\left(\rho(u)\rho(v) + \rho(v)\rho(u)\right) = -\underset{\uparrow \text{ metric on } T^*M}{g(u,v)}\, \mathbb{1}.$$

Th: D_ρ is a Dirac operator $\Leftrightarrow$ the above condition holds.

From $\rho_x : T_x^*M \to \underset{\text{ℝ-alg., endowed with composition.}}{\mathrm{End}(E_x)}$, construct the tensor version

$\rho_x^\otimes : \mathrm{Tens}(T_x^*M) \longrightarrow \mathrm{End}(E_x)$, which sends $u_1 \otimes \cdots \otimes u_\ell$ to $\rho(u_1) \circ \cdots \circ \rho(u_\ell)$.

$\mathrm{Tens}(T_x^*M) = \bigcup_{m \geq 0} (T_x^*M)^{\otimes m}$

If ρ satisfies the above condition (i.e. D_ρ is Dirac) then $\rho_x^{\otimes m}$ factors through $\langle u \otimes v + v \otimes u + 2g(u,v)\underset{\uparrow \text{ unit of } \mathrm{Tens}(T^*M)}{1} \rangle$:

$$\rho_x^\otimes : \mathrm{Tens}(T_x^*M) \big/ \langle u\otimes v + v \otimes u + 2g(u,v)1, (u,v) \in T_x^*M^2 \rangle \longrightarrow \mathrm{End}(E_x)$$

Def: For (V,g) a euclidean space*, the Clifford algebra $Cl(V,g)$:

$$\boxed{Cl(V,g) := \mathrm{Tens}(V) \big/ \langle u \otimes v + v \otimes u + 2g(u,v)1, (u,v) \in V^2 \rangle}$$

We can paste this construction to get an ℝ-algebra bundle:

$$\rho^\otimes : Cl(T^*M, g) \longrightarrow \mathrm{End}(E)$$

In other words, $\rho^\otimes$ is a collection of representations of $Cl(T_x^*M, g)$.

* More generally, (V,g) is a vector space with a non-degenerate quad. form

Rk: If $\rho : Cl(\mathbb{R}^m, \text{eucl}) \curvearrowright V$ is an action on a vector space (for instance V is   , eucl) and the action is left-multiplication) then the operator

basis of $\mathbb{R}^m$

$$\sum_i \rho(e_i) \frac{\partial}{\partial x_i},$$

partial derivatives $\mathcal{C}^\infty(\mathbb{R}^m, E) \to \mathcal{C}^\infty(\mathbb{R}^m, E)$

also denoted

$$\sum_i e_i \cdot \frac{\partial}{\partial x_i},$$

is a Dirac operator on the fibre bundle $\begin{matrix} \mathbb{R}^m \times V \\ \downarrow \\ \mathbb{R}^m \end{matrix}$.

In some philosophical sense, the study of Dirac operators reduces to three main problems:

Understanding the representations of $Cl(\mathbb{R}^m, \text{eucl})$ $\longleftrightarrow$ Understanding the bundle $Cl(T^*M, g)$

Understanding their relationship

Twisted Dirac operators.

Underlined words are "defined" in the following part.

On certain Riemannian manifolds, the so-called spin manifolds, there is a canonical bundle called the spinor bundle S. It comes with a connection ∇^S and an action of $Cl(T^*M, g)$ locally modelled on the spin representation $Cl(\mathbb{R}^m, \text{eucl}) \curvearrowright \Delta_m$. We can then define the Dirac operator D_ρ, commonly denoted $\not{D}$. More generally, if E is vector bundle with a connection ∇^E, the bundle $S \otimes E$ inherits a connection $\nabla^{S\otimes E}$ defined by $\nabla^S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes \nabla^E$, and the action of $Cl(T^*M, g)$ on the first factor makes it possible to define $\not{D}^E = D_{\rho \otimes \mathbb{1}}$, which we call the twisted Dirac operator associated to E. These twisted Dirac operators are more or less the building blocks from which we construct every operator coming from the index theorem.

They are universal in the following sense. $Cl(V, \text{eucl})$ comes with a $\mathbb{Z}_2$-grading inherited from $\text{Tens}^{odd}(V) \oplus \text{Tens}^{even}(V)$. If (M, g) is an even-dimensional spin manifold and E is a $Cl(T^*M, g)$-module, if moreover $E = E^+ \oplus E^-$ is $\mathbb{Z}_2$-graded and the action is graded as well, then E is in fact a twisted bundle

$\underset{Cl^+}{w} \cdot \underset{E^\pm}{e} \in E^\pm$ and $\underset{Cl^-}{w} \cdot \underset{E^\pm}{e} \in E^\mp$

$E \simeq S \otimes E'$.

S itself is $\mathbb{Z}_2$-graded, and has a Hermitian structure. Using the identifications $(S^\pm)^* \simeq S^\pm$, we can write $\not{D} = \begin{pmatrix} 0 & \not{\partial}^* \\ \not{\partial} & 0 \end{pmatrix}$ for

S^+ and S^- are orthogonal

some $\not{\partial} : \mathcal{C}^\infty(S^+) \to \mathcal{C}^\infty(S^-)$. Similarly, $\not{D}^E = \begin{pmatrix} 0 & \not{\partial}^* \otimes \mathbb{1}_E \\ \not{\partial} \otimes \mathbb{1}_E & 0 \end{pmatrix}$

$(S \otimes E)^+ = S^+ \otimes E$

$(S \otimes E)^- = S^- \otimes E$

a. **Def:** Spin(n) is the only connected 2-fold cover of SO(n). It fits in the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 0.$$

A Riemannian* manifold (M,g) is Spin if there exists a Spin(n) principal bundle $\begin{matrix} P_{spin} \\ \downarrow \\ M \end{matrix}$ such that the following "diagram of actions" commutes:

*+ oriented

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, phantom, "\curvearrowright"] \arrow[d, "\text{2-fold cover}"'] & P_{spin} \arrow[d, "\text{Some bundle morphism}"] \\ SO(n) \arrow[r, phantom, "\curvearrowright"] & P_{SO} \end{tikzcd}$$

right-action coming from the principal bundle structure.

bundle of oriented orthonormal frames.

Such a bundle exists if and only if the second Stiefel-Whitney class $w_2(M)$ vanishes ($w_1(M)=0$ since M is oriented).

b. Let q be the quadratic form on $\mathbb{C}^{2n}$ defined by the matrix

$$\begin{array}{c} \left(\begin{array}{c|c} 0 & 1 \ddots 1 \\ \hline 1 \ddots 1 & 0 \end{array}\right) \\ \underbrace{\qquad}_{W}\underbrace{\qquad}_{W^*} \end{array}.$$

Then $\mathbb{C}^{2n} \simeq \mathbb{C}^n \oplus \mathbb{C}^n = W \oplus W^*$, where the dual is identified with the initial space via q. $\mathbb{C}^{2n}$ acts on $\Lambda^* W$ via

$$\frac{1}{\sqrt{2}}\, v \cdot (w_1 \wedge \dots \wedge w_k) = w \wedge w_1 \wedge \dots \wedge w_k + w^*(w_1)\, w_2 \wedge \dots \wedge w_k.$$

$\mathbb{C}^{2n} \ni v = w + w^* \in W \oplus W^*$

This action extends to an action $Cl(\mathbb{C}^{2n}, q) \curvearrowright \Lambda^* W$, and it is graded. It is actually an isomorphism: $Cl(\mathbb{C}^{2n}, q) \simeq \mathrm{End}(\Lambda^* W)$ and the representation is irreducible.

from $\Lambda^* = \Lambda^{even} + \Lambda^{odd}$

b. continued If we consider $Cl(\mathbb{R}^{2n}, q)$, $Cl(\mathbb{C}^{2n+1}, q)$ or $Cl(\mathbb{R}^{2n+1}, q)$, the situation is more complicated. Nevertheless, we construct in a similar fashion the spin (or spinor) representations $Cl(\mathbb{R}^n, eucl) \curvearrowright \Delta_n$.

c. There exists in $Cl(\mathbb{R}^n, eucl)$ a copy of Spin(n): it is the set

$$\mathrm{Spin}(n) \simeq \{ u_1 \cdots u_{2k},\ k \in \mathbb{N},\ u_i \in \mathbb{R}^n \subset Cl(\mathbb{R}^n, eucl),\ \|u_i\| = 1 \}.$$

In particular, if (M,g) is a spin manifold with Spin(n)-bundle P_{spin}, then the action of Spin(n) on Δ_n can be used to define the associated bundle $P_{spin} \times_{spin} \Delta_n$. This is the spinor bundle S.

Moreover, there is an action of Spin(n) on $Cl(\mathbb{R}^n, eucl)$ (get from Spin(n) to SO(n), lift the standard action $SO(n) \curvearrowright \mathbb{R}^n$ to an action on $\mathrm{Tens}(\mathbb{R}^n)$, take the quotient; this is not left-multiplication) such that $Cl(T^*M, g) \simeq P_{Spin} \times_{Spin} Cl(\mathbb{R}^n, eucl)$.

An easy way to define an action of $Cl(T^*M, g)$ on S would be to set $(g, u) \cdot (g, s) := (g, u \cdot s)$ and to take the quotient. To

in $P_{Spin} \times Cl(\mathbb{R}^n, eucl)$ — in $P_{Spin} \times \Delta_n$

ensure that this makes sense, we need $g \cdot (u \cdot s) = (g \cdot u) \cdot (g \cdot s)$ for all $\begin{cases} g \in \mathrm{Spin}(n), \\ u \in Cl(\mathbb{R}^n, eucl), \\ s \in \Delta_n. \end{cases}$ Luckily, everything was defined in order to make it work, and S is indeed a $Cl(T^*M, g)$-module. Since M is Riemannian it comes with the Levi-Civita connection, from which we get a connection on P_{Spin}, then ∇^S on S. Spin(n) acts on Δ_n by isometries, so S has a metric coming from Δ_n. Finally, Δ_n is graded and $\mathrm{Spin}(n) \subset Cl^+(\mathbb{R}^n, eucl)$ so the action preserves the grading; S is graded and the action of a vector in $T^*M \subset Cl(T^*M, g)$ exchanges S^+ with S^-.